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Research Article

Enhanced Squeezing and Entanglement in Nondegenerate Three-Level Laser Coupled to Squeezed Vacuum Reservoir

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Squeezing and entanglement of a two-mode cascade laser, produced by a three-level atom which is initially prepared by a coherent superposition of the top and bottom levels then injected into a cavity coupled to a two-mode squeezed vacuum reservoir is discussed. I obtain stochastic differential equations associated with the normal ordering using the pertinent master equation. Making use of the solutions of the resulting differential equations, we determined the mean photon number for the cavity mode and their correlation, EPR variables, smallest eigenvalue of the symplectic matrix, intensity difference fluctuation, and photon number correlation. It is found that the squeezed vacuum reservoir increases the degree of the statistical and nonclassical features of light produced by the system. Furthermore, using the criteria developed by logarithm negativity and Hillery-Zubairy criteria, the quantum entanglement of the cavity mode is quantified. It is found that the degree of the entanglement for the system under consideration increases with the squeezing parameter of the squeezed vacuum reservoir.

1. Introduction

The ladder-type three-level laser has received a considerable interest over the years in light of its potential application as a source of radiation with various quantum properties [1–8]. The quantum properties of the light, in this device, is attributed to atomic coherence that can be induced either by preparing the atoms initially in a coherent superposition of the top and bottom levels [7, 8] or coupling these levels by an external radiation [9–11] or using these mechanisms together [12].

A three-level laser with a coherent superposition of the top and bottom levels of the injected atoms has been studied by different authors [13–39]. This study shows that a quantum optical system can generate light in squeezed state under certain conditions. Tesfa [40] has studied the entanglement amplification and squeezing properties of the cavity mode produced by nondegenerate three-level laser applying the solution of the stochastic differential equation when the atomic coherence is introduced initially preparing a three-level atom by a coherent superposition of the top and bottom levels via the intermediate. He showed that the two-mode cavity radiation exhibits squeezing properties under certain conditions pertaining to the initial preparation of the superposition, where the degree of the squeezing increases with the linear gain coefficient. In particular, the squeezing properties exist if the atoms are initially prepared in such a way that there are more atoms in the bottom level than in the upper level. A relatively better squeezing is found when a sufficiently large number of atoms are injected into the cavity and when the atoms are initially prepared with nearly 48% probability to be in the top level and with a significant entanglement between the states of light generated in the cavity of the nondegenerate three-level cascade laser, due to the strong correlation between the radiation emitted when the atom decays from the top level to the bottom level via the intermediate level.

Villas-Bôas and Moussa [41] showed that a single driven nondegenerate three-level atom in cascade configuration which is initially prepared in the coherent superposition of the top and bottom levels and placed in the cavity can be used to generate the superposition of a highly squeezed two-mode radiation in the weak driving, and they also found that squeezing is relatively better in the strong driving limit.

Furthermore, Abebe and Feyisa [42] studied the dynamics of a nondegenerate three-level laser with a parametric amplifier and coupled to a two-mode squeezed vacuum reservoir, and they showed that a large amplitude of the classical driving radiation induces a strong correlation between the top and bottom states of three-level atoms to produce a high degree of squeezing and entanglement. Moreover, the presence of a parametric amplifier and squeezed parameter is found to enhance the degree of squeezing and entanglement of the cavity light.

In this paper, we analyze the dynamics of a nondegenerate three-level laser coupled to a two-mode squeezed vacuum reservoir via a single port mirror in the absence of the parametric amplifier. In the presence of the parametric amplifier, the dynamics of a nondegenerate three-level laser coupled to the squeezed vacuum reservoir was studied by [42], the significance of the squeezing parameter and parametric amplifier on the degree of squeezing and entanglement was discussed, and the squeezing parameter exhibits the same behavior which agrees with my study in enhancing the degree of squeezing and entanglement in my study. The photon statistics and nonclassical properties of the generated cavity radiation is studied; in order to carry out our analyses, we first derive the master equation in the good cavity limit, under the consideration of linear and adiabatic approximations, then employing the master equation, we determined the stochastic differential equations, solutions for c-number cavity mode variables, and correlation property of the noise forces associated with the normal ordering. Using the resulting solutions, the mean photon number, intensity difference fluctuation, and photon number correlation of the cavity radiations are obtained; also, the effects of the squeezing parameter on the nonclassical and statistical property of light are discussed. Moreover, employing the criteria developed for continuous variables such as logarithmic negativity and Hillery-Zubairy criteria, the nonclassical property of light (entanglement) produced by the system is quantified.

2. The Model and Hamiltonian

Here, we want to drive the master equation for a nondegenerate three-level laser with the cavity modes driven by a two-mode coherent light and coupled to a two-mode squeezed vacuum reservoir.

We represent the top, intermediate, and bottom levels of a three-level atom in a cascade configuration by $|a\rangle$, $|b\rangle$, and $|c\rangle$, respectively, as shown in Figure 1. In addition, we assume the two modes a and b to be at resonance with the two transitions $|a\rangle \rightarrow |c\rangle$ and $|b\rangle \rightarrow |c\rangle$, respectively, and the direct transition between level $|a\rangle$ and level $|c\rangle$ to be dipole forbidden, and also, we consider the case in which three-level atoms in the cascade configuration and initially prepared in a coherent superposition of the top and bottom levels are injected into a cavity at a constant rate r_a (rate of atomic injection in to the cavity) and removed after some time τ, which is long enough for the atoms to decay spontaneously to levels other than the middle level or the lower level.

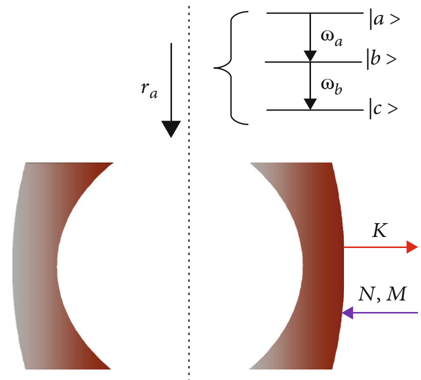

Figure 1: Schematic representation of a nondegenerate three-level laser coupled to squeezed vacuum reservoir.

The spontaneous decay rate γ is assumed to be the same for the top and intermediate levels. Three-level atoms resonantly interact with the cavity modes and classical pumping radiation in the laser cavity. In the good cavity limit, $\kappa < < \gamma$, where κ is the cavity damping rate; the cavity-mode variables change slowly compared with the atomic variables. Hence, the atomic variables will reach steady state in a relatively short time. The time derivative of such variables can then be set to zero, while keeping the remaining terms at time t. This procedure is referred to as the adiabatic approximation scheme. Since the coupling constant is supposed to be small, a linear analysis that amounts to dropping the higher order terms in g is employed. It is good to note that the linear approximation preserves the quantum properties we seek to study as these properties are attributed to the classical driving radiation that couples the top and bottom states. When an atom makes a transition between the top and bottom levels via the intermediate level, two correlated photon of nondegenerate frequencies, ω_a and ω_b, are generated. We assume that these transition frequencies are at resonance with the two nondegenerate cavity modes. Moreover, the cavity mode interacts with the squeezed vacuum reservoir. This system is plotted in Figure 1.

The quantum optical system outlined in Figure 1 can be described in the interaction picture by the Hamiltonian

$$\hat{H} = \hat{H}_I + \hat{H}_{SR}, \tag{1}$$

where

$$\hat{H}_I = ig\left[a\wedge^{\dagger} \mid b\rangle\left\langle a\right|+b\wedge^{\dagger} \mid c\left.\right\rangle\langle b|-|\hat{a}\rangle\langle b \mid \hat{a}-|b\rangle \left\langle c \mid \hat{b}\right.\right], \tag{2}$$

is the Hamiltonian describing the interactions of the three-level atom with the cavity mode, g is the atom-cavity mode coupling constant assumed to be the same for both transitions, and $\hat{a}$ and $\hat{b}$ are annihilation operators for the two cavity modes.

Similarly, the Hamiltonian describing the interaction of the cavity mode and squeezed vacuum reservoir is given as

$$\hat{H}_{SR}(t) = i\sum_m \lambda_m \left(a\wedge^\dagger \hat{C}_m e^{i(\omega_a-\omega_m)t} - \hat{a}\hat{C}_m^\dagger e^{-i(\omega_a-\omega_m)t} \right) + i\sum_n \lambda_n \left(b\wedge^\dagger \hat{D}_n e^{i(\omega_b-\omega_n)t} - \hat{b}\hat{D}_n^\dagger e^{-i(\omega_b-\omega_n)t} \right), \quad (3)$$

where $\hat{a}$ and $\hat{b}$ are the annihilation operator for the cavity modes with the frequencies ω_a and ω_b and C_m and D_n are the annihilation operator for the reservoir modes having the frequencies ω_m and ω_n. λ_m and λ_n are the coupling constants between the cavity modes and the reservoir modes. Here, we take the initial state of a single three-level atom considered to be

$$|\psi(0)\rangle = C_a(0)|a\rangle + C_c(0)|c\rangle, \quad (4)$$

the corresponding initial density operator is given as

$$\rho^{(0)} = \rho_{aa}^{(0)}|a\rangle\Big\langle a| + \rho_{ac}^{(0)} \mid a\Big\rangle\Big\langle c| + \rho_{ca}^{(0)} \mid c\Big\rangle\Big\langle a| + \rho_{cc}^{(0)} \mid c\Big\rangle\langle c|, \quad (5)$$

where $\rho_{aa}^{(0)} = |C_a(0)|^2$ and $\rho_{cc}^{(0)} = |C_c(0)|^2$ are the probability for the atom to be in the upper and lower levels at the initial time and

$$\rho_{ac}^{(0)} = C_a(0)C_c^*(0), \rho_{ca}^{(0)} = C_c(0)C_a^*(0), \quad (6)$$

represents the atomic coherence at the initial time. The master equation corresponding to Equation (2) is

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = {} & \frac{1}{2}A\rho_{aa}^{(0)}\left[2a\wedge^\dagger\hat{\rho}\hat{a} - \hat{a}a\wedge^\dagger\hat{\rho} - \hat{\rho}\hat{a}a\wedge^\dagger\right] \\ & + \frac{1}{2}A\rho_{cc}^{(0)}\left[2\hat{b}\hat{\rho}b\wedge^\dagger - b\wedge^\dagger\hat{b}\hat{\rho} - \hat{\rho}b\wedge^\dagger\hat{b}\right] \\ & + \frac{\kappa}{2}\left(2\hat{a}\hat{\rho}a\wedge^\dagger - a\wedge^\dagger\hat{a}\hat{\rho} - \hat{\rho}a\wedge^\dagger\hat{a}\right) \\ & + \frac{\kappa}{2}\left(2\hat{b}\hat{\rho}b\wedge^\dagger - b\wedge^\dagger\hat{b}\hat{\rho} - \hat{\rho}b\wedge^\dagger\hat{b}\right) \\ & - \frac{1}{2}A\rho_{ac}^{(0)}\left(2\hat{b}\hat{\rho}\hat{a} - \hat{\rho}\hat{a}\hat{b} - \hat{a}\hat{b}\hat{\rho}\right) \\ & - \frac{1}{2}A\rho_{ca}^{(0)}\left(2a\wedge^\dagger\hat{\rho}b\wedge^\dagger - \hat{\rho}a\wedge^\dagger b\wedge^\dagger - a\wedge^\dagger b\wedge^\dagger\hat{\rho}\right),\end{aligned} \quad (7)$$

where $A = 2g^2 r_a/\gamma^2$ is the linear gain coefficient.

The master equation resulted from the interaction of the cavity mode and a two-mode squeezed vacuum reservoir is

$$\begin{aligned}\frac{d}{dt}\hat{\rho}(t) = {} & -i\left[\hat{H}_{SR}(t), \hat{\rho}(t)\right] + \frac{\kappa}{2}\left(2a\wedge^\dagger\hat{\rho}\hat{a} - \hat{a}a\wedge^\dagger\hat{\rho} - \hat{\rho}\hat{a}a\wedge^\dagger\right) \\ & + \frac{\kappa}{2}\left(2b\wedge^\dagger\hat{\rho}\hat{b} - \hat{b}b\wedge^\dagger\hat{\rho} - \hat{\rho}\hat{b}b\wedge^\dagger\right) \\ & + \frac{\kappa}{2}(N+1)\left(2\hat{a}\hat{\rho}a\wedge^\dagger - a\wedge^\dagger\hat{a}\hat{\rho} - \hat{\rho}a\wedge^\dagger\hat{a}\right) \\ & + \frac{\kappa}{2}(N+1)\left(2\hat{b}\hat{\rho}b\wedge^\dagger - b\wedge^\dagger\hat{b}\hat{\rho} - \hat{\rho}b\wedge^\dagger\hat{b}\right) \\ & + \kappa M\left(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - \hat{b}\hat{\rho}\hat{a} - \hat{a}\hat{\rho}\hat{b}\right) \\ & + \kappa M\left(\hat{\rho}a\wedge^\dagger b\wedge^\dagger + a\wedge^\dagger b\wedge^\dagger\hat{\rho} - b\wedge^\dagger\hat{\rho}a\wedge^\dagger - a\wedge^\dagger\hat{\rho}b\wedge^\dagger\right).\end{aligned} \quad (8)$$

Finally, substituting Equation (3) into Equation (8), we obtain the master equation of a nondegenerate three-level laser coupled to a two-mode squeezed vacuum reservoir as

$$\begin{aligned}\frac{d\hat{\rho}(t)}{dt} = {} & \frac{\kappa}{2}(N+1)\left(2\hat{a}\hat{\rho}a\wedge^\dagger - a\wedge^\dagger\hat{a}\hat{\rho} - \hat{\rho}a\wedge^\dagger\hat{a}\right) \\ & + \frac{1}{2}\left[A\rho_{cc}^{(0)} + \kappa(N+1)\right]\left(2\hat{b}\hat{\rho}b\wedge^\dagger - b\wedge^\dagger\hat{b}\hat{\rho} - \hat{\rho}b\wedge^\dagger\hat{b}\right) \\ & + \frac{1}{2}\left(A\rho_{aa}^{(0)} + \kappa N\right)\left(2a\wedge^\dagger\hat{\rho}\hat{a} - \hat{a}a\wedge^\dagger\hat{\rho} - \hat{\rho}\hat{a}a\wedge^\dagger\right) \\ & + \frac{1}{2}\kappa N\left(2b\wedge^\dagger\hat{\rho}\hat{b} - \hat{b}b\wedge^\dagger\hat{\rho} - \hat{\rho}\hat{b}b\wedge^\dagger\right) \\ & + \frac{1}{2}\left(A\rho_{ac}^{(0)} + \kappa M\right)\left(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{b}\hat{\rho}\hat{a}\right) \\ & + \frac{\kappa M}{2}\left(\hat{\rho}\hat{a}\hat{b} + \hat{a}\hat{b}\hat{\rho} - 2\hat{a}\hat{\rho}\hat{b}\right) \\ & + \frac{1}{2}\left(A\rho_{ac}^{(0)} + \kappa M\right)\left(\hat{\rho}a\wedge^\dagger b\wedge^\dagger + a\wedge^\dagger b\wedge^\dagger\hat{\rho} - 2a\wedge^\dagger\hat{\rho}b\wedge^\dagger\right) \\ & + \frac{\kappa M}{2}\left(\hat{\rho}a\wedge^\dagger b\wedge^\dagger + a\wedge^\dagger b\wedge^\dagger\hat{\rho} - 2b\wedge^\dagger\hat{\rho}a\wedge^\dagger\right).\end{aligned} \quad (9)$$

Equation (9) indicates the stochastic master equation which contains all necessary information regarding the dynamics of the system involving the effect of the huge external environment. Moreover, $A = 2r_a g/\gamma^2$ represents the rate of injecting atoms which are initially prepared at the bottom level. The constants N and M, which describe the effect of the external environment, are related to each other through the squeeze parameter r as $M = \sqrt{N(N+1)}$ in which $N = \sin hr$. Hence, the squeeze parameter quantifies the mean photon number of the two-mode squeezed vacuum reservoir and intermodal correlations among the reservoir submodes.

Employing the master equation, the time development of the c-number cavity mode variables, $\alpha(t)$ and $\beta(t)$, associated with the normal ordering, can be put in the form

$$\frac{d}{dt}\langle\alpha\rangle = -\frac{\mu_\alpha}{2}\langle\alpha\rangle - \frac{\rho_{ac}^{(0)}}{2}\langle\beta^*\rangle + f_\alpha, \quad (10)$$

$$\frac{d}{dt}\langle\beta\rangle = -\frac{\mu_b}{2}\langle\beta\rangle + \frac{\rho_{ac}^{(0)}}{2}\langle\alpha^*\rangle + f_\beta, \tag{11}$$

where

$$\mu_a = \kappa - A\rho_{aa}^{(0)}, \tag{12}$$

$$\mu_b = \kappa + A\rho_{cc}^{(0)}, \tag{13}$$

for f_α and f_β are the pertinent noise forces associated with the fluctuation of the external environment. Making use of Equations (10) and (11), the correlation properties of the noise forces can be readily put as

$$\begin{aligned}
&\left\langle f_\alpha\left(t'\right) f_\alpha(t)\right\rangle = 0,\\
&\left\langle f_\beta\left(t'\right) f_\beta(t)\right\rangle = \left\langle f_\beta^*\left(t'\right) f_\beta^*(t)\right\rangle = 0,\\
&\left\langle f_\alpha^*(t) f_\alpha\left(t'\right)\right\rangle = \left\langle f_\alpha^*\left(t'\right) f_\alpha(t)\right\rangle = \left(A\rho_{aa}^{(0)} + \kappa N\right)\delta\left(t-t'\right),\\
&\left\langle f_\beta^*(t) f_\beta\left(t'\right)\right\rangle = \left\langle f_\beta^*\left(t'\right) f_\beta(t)\right\rangle = \kappa N\delta\left(t-t'\right),\\
&\left\langle f_\alpha(t) f_\beta\left(t'\right)\right\rangle = \left\langle f_\alpha\left(t'\right) f_\beta(t)\right\rangle = \left(\frac{1}{2}A\rho_{ac}^{(0)} + \kappa M\right)\delta\left(t-t'\right),\\
&\left\langle f_\alpha(t) f_\beta^*\left(t'\right)\right\rangle = \left\langle f_\alpha\left(t'\right) f_\beta^*(t)\right\rangle = 0,\\
&\left\langle f_\beta^*\left(t'\right) f_\alpha(t)\right\rangle = \left\langle f_\beta^* f_\alpha\left(t'\right)(t)\right\rangle = 0,\\
&\left\langle f_\alpha(t) f_\alpha^*\left(t'\right)\right\rangle = \left\langle f_\alpha\left(t'\right) f_\alpha^*(t)\right\rangle = \kappa(N+1)\delta\left(t-t'\right).
\end{aligned} \tag{14}$$

The solution of Equations (10) and (11) is obtained as follows:

$$\begin{aligned}
\alpha(t) &= \alpha(0)e^{-\mu_a t/2} - \int_0^t e^{-\mu_a\left(t-t'\right)/2}\left(\frac{1}{2}A\rho_{ac}^{(0)}\alpha^*\left(t'\right) + f_\alpha\left(t'\right)\right)dt',\\
\beta(t) &= \beta(0)e^{-\mu_b t/2} + \int_0^t e^{-\mu_b\left(t-t'\right)/2}\left(\frac{1}{2}A\rho_{ac}^{(0)}\beta^*\left(t'\right) + f_\beta\left(t'\right)\right)dt',
\end{aligned} \tag{15}$$

where the value of μ_a and μ_b are stated by Equations (12) and (13), respectively.

3. Quadrature Variance

In this section, we seek to study the quadrature squeezing of the light produced by a nondegenerate three-level laser coupled to a two-mode squeezed vacuum reservoir via a single-port mirror. In general, the squeezing properties of a two-mode cavity radiation can be described by two quadrature operators of the cavity mode operator.

$$\widehat{c}_+ = \left(c\wedge^\dagger + \widehat{c}\right), \tag{16}$$

$$\widehat{c}_- = i\left(c\wedge^\dagger - \widehat{c}\right), \tag{17}$$

in which $\widehat{c} = (\widehat{a}_1 + \widehat{a}_2)/\sqrt{2}$ with $\widehat{a}_1$ and $\widehat{a}_2$ represent the separate modes of cavity light emitted from the three-level atoms. Employing the commutation relation $[\widehat{c}, c\wedge^\dagger] = 1$, the Hermitian and noncommuting quadrature operators, $\widehat{c}_+$ and $\widehat{c}_-$, satisfy the relation

$$[\widehat{c}_+, \widehat{c}_-] = 2i. \tag{18}$$

On the basis of these definitions, a two-mode light is said to be in a squeezed state if either $\Delta c_+^2 < 1$ and $\Delta c_-^2 > 1$ or $\Delta c_+^2 > 1$ and $\Delta c_-^2 < 1$, such that $\Delta c_+ \Delta c_- \geq 1$. The variances of the quadrature operators can be expressed as

$$\Delta c_\pm^2 = \left\langle \widehat{c}_\pm^2\right\rangle - \langle c\wedge_\pm\rangle^2. \tag{19}$$

It is then obvious that for the combined system, the squeezing occurs in the minus quadrature. In order to clearly see the effect of the two-squeezed vacuum reservoir on the degree of squeezing of the two-mode light generated by the laser system, we plot, in Figure 2, Equation (25) versus η for different values of the squeeze parameter r. This figure indicates that the two-mode squeezed vacuum reservoir considerably increases the amount of two-mode squeezing in the cavity for relatively small values of η. For instance, in the absence of the squeezed vacuum reservoir and for $A = 100$ and $\kappa = 0.8$, the amount of squeezing is found to be 65.33% below the vacuum level. However, in the presence of the squeezed vacuum reservoir with $r = 0.5$ and for the same parameters used above, the amount of squeezing is calculated to be 85.3%. Hence, with this choice of squeeze parameter, the degree of squeezing of the two-mode light is enhanced by over 19.97%. When the squeeze parameter r increases, the value of g at which the maximum squeezing occurs approaches to zero. In addition, as r increases, the squeezing decreases for values of η close to one and even disappears for values of η very close to one. Furthermore, in Figure 3, we plot the variance of the minus quadrature of the two-mode light versus A and squeeze parameter r. It is easy to see from this plot that the system is in the squeezed state for all values of considered variables and squeezing increases with r and linear gain coefficient in general. It is possible to express the variance of the quadrature operators (16) and (17), in terms of the c-number variables associated with the normal ordering taking the cavity modes to be initially in a two-mode squeezed vacuum state, as

$$\Delta c_\pm^2 = 1 + \langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle \pm 2\langle\alpha(t)\beta(t)\rangle. \tag{20}$$

To this effect, assuming the initial states of the cavity modes to be in a vacuum state, and taking into account the fact that the noise force at some time t does not affect the cavity mode variables at earlier times, it can be verified that

$$\left\langle\alpha^2\right\rangle = \left\langle\beta^2\right\rangle = \langle\alpha^*\beta\rangle = 0, \tag{21}$$

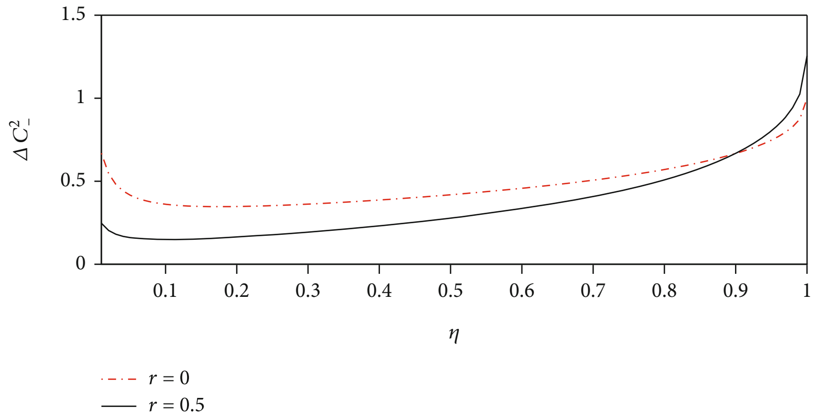

Figure 2: Plots of minus quadrature variances (Equation (25)) versus η for $A = 100$ and $\kappa = 0.8$ and for different values of squeeze parameter.

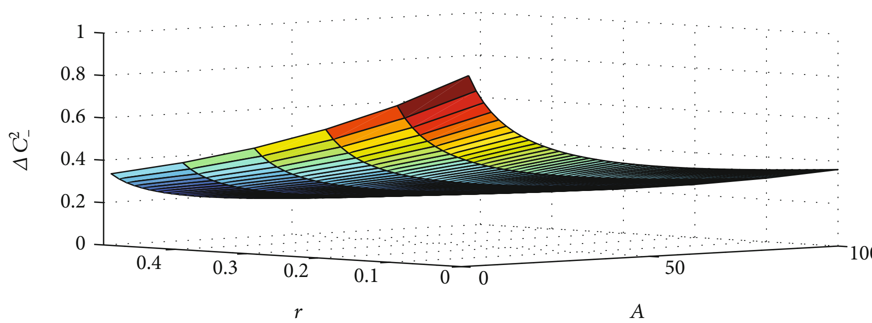

Figure 3: Plots of minus quadrature variances (Equation (25)) versus r and A, for $\kappa = 0.8$ and $\eta = 0.1$.

$$\langle \alpha^*(t)\alpha(t)\rangle = \frac{\kappa A(1-\eta)(4\kappa + 3A\eta + A)}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)} + \frac{\left[2\kappa(2\kappa + 2A\eta + A)A^2(1+\eta)\right]2\kappa N}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)} - \frac{\left[A\sqrt{1-\eta^2}(2\kappa + A\eta + A)\right]2\kappa M}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}. \tag{22}$$

Similarly, we get

$$\langle \beta^*(t)\beta(t)\rangle = \frac{\left[\kappa A^2(1-\eta^2) + \left(A\sqrt{1-\eta^2}\right)(2\kappa + A\eta - A)\right]2\kappa M}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)} + \frac{\left[2\kappa(2\kappa + 2A\eta - A) + A^2(1-\eta)\right]2\kappa N}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}, \tag{23}$$

$$\langle \alpha(t)\beta(t)\rangle = \frac{\left[A\kappa\sqrt{1-\eta^2}(2\kappa + A\eta + A) + (2\kappa + A\eta)^2 - A^2\right]2\kappa M}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)} + \frac{\left[A^2\sqrt{1-\eta^2}\right]2\kappa N}{4\kappa(\kappa + A\eta)(2\kappa + A\eta)}. \tag{24}$$

By inserting Equations (21)–(24) into Equation (20), we get

$$\Delta c_-^2 = 1 + \frac{\kappa A(1-\eta)(2\kappa + 2A\eta + A) - 2\kappa A^2\eta^2 N}{2[\kappa(\kappa + A\eta)(2\kappa + A\eta)]} + \frac{\kappa A\sqrt{1-\eta^2}(2\kappa + A\eta + A)}{2[\kappa(\kappa + A\eta)(2\kappa + A\eta)]} + \frac{\kappa\left[2\kappa + A\eta(2\kappa + A\eta)(N+M) + A^2\left(1 + \sqrt{1-\eta^2}\right)(N-M)\right]}{[\kappa(\kappa + A\eta)(2\kappa + A\eta)]}. \tag{25}$$

The result obtained in Equation (25) represents the steady state solution of Δc_-^2 quantum optical system.

4. Photon Statistics

4.1. Mean Photon Number. In order to learn about the brightness of the generated light and its relation with entanglement, it is worthwhile to study the mean number of photon of the two-mode cavity radiation. In terms of the annihilation operator of the cavity radiation, the mean photon number of the cavity light can be defined as

$$\bar{n} = \left\langle c\wedge^{\dagger}(t)\widehat{c}(t)\right\rangle, \tag{26}$$

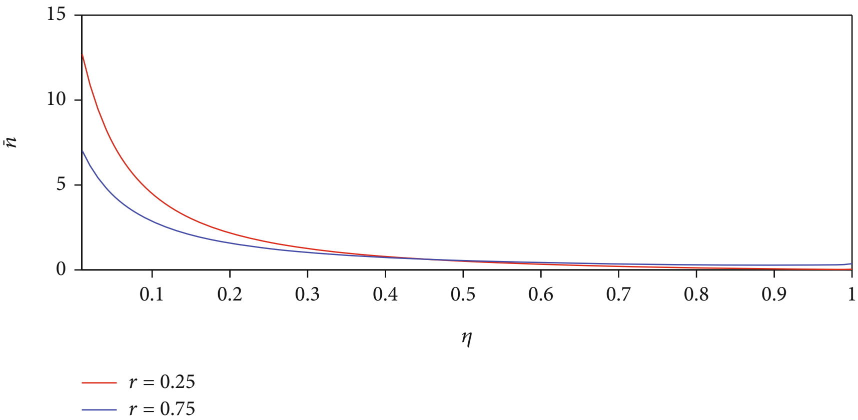

Figure 4: Plots of the mean photon number (Equation (27)) versus η for $A = 100$ and $\kappa = 0.8$ and for different values of squeeze parameter.

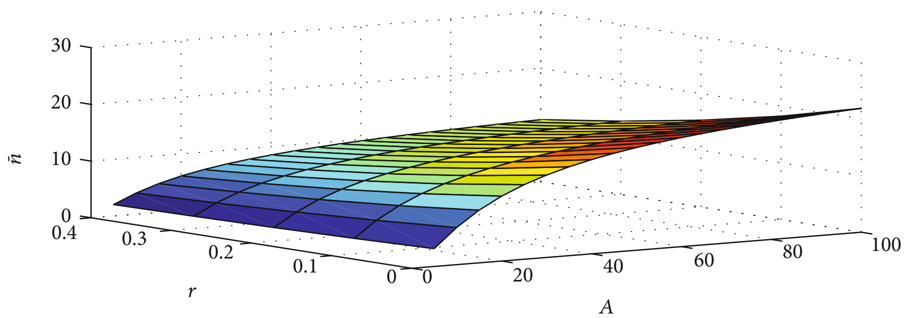

Figure 5: Plots of the mean photon number (Equation (28)) versus r and A for $\eta = 0.1$ and $\kappa = 0.8$.

The operators in Equation (26) are already in the normal order; it is possible to put this expression in terms of c-number variables associated with the normal ordering in the form

$$\bar{n} = \frac{1}{2}\left(\langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle + \langle \alpha^*(t)\beta(t)\rangle + \langle \beta^*(t)\alpha(t)\rangle\right). \quad (27)$$

In Figure 4, we plot the mean photon number of the two-mode light versus η for different values of the squeezed vacuum reservoir; it is very easy to see from the figure that the squeezed vacuum increases the mean photon number in the region where there is strong squeezing and entanglement. Hence, this system generates a bright and highly squeezed as well as entangled light. We also notice that the mean number of photons is larger for small values of η at which the squeezing is found to be relatively higher; we obtain $\bar{n} = 7.022$ for $A = 100$, $\kappa = 0.8$, $r = 0.75$, and $\eta = 0.01$ at which the squeezing is found to be maximum.

In view of Equation (21), the above Equation (26) is reduced to

$$\bar{n} = \frac{1}{2}\langle \alpha^*(t)\alpha(t)\rangle + \langle \beta^*(t)\beta(t)\rangle. \quad (28)$$

Equation (28) represents the mean photon number pair of the system.

One can see from Figure 5 that the mean number of the photon decreases with the squeeze parameter for certain values of the squeeze parameter and η. We also found that a similar situation exists except for very small values of the linear gain coefficient, where the mean number of the photon increases with r for all values of η. The mean number of the photon is found to be $\bar{n} = 9.38$ for $r = 0.5$, $\eta = 0.1$, and $A = 100$ where the degree of squeezing is maximum.

4.2. Intensity Difference Fluctuation. The variance of the intensity difference can be defined as

$$\Delta\hat{I}_D = \left\langle \hat{I}_D^2 \right\rangle - \langle I\wedge_D\rangle^2, \quad (29)$$

where the intensity difference is given as

$$\hat{I}_D = a\wedge^{\dagger}(t)\hat{a}(t) - b\wedge^{\dagger}(t)\hat{b}(t). \quad (30)$$

Substituting Equation (45) into (44) and rewriting in a more convenient way, we obtain

$$\begin{aligned}\hat{I}_D^2 &= \langle \alpha^*(t)\alpha(t)\rangle[1 + \langle \alpha^*(t)\alpha(t)\rangle] + \langle \beta^*(t)\beta(t)\rangle \\ &\quad \times [1 + \langle \beta^*(t)\beta(t)\rangle] - 2\langle \alpha(t)\beta(t)\rangle^2.\end{aligned} \quad (31)$$

We can understand that Figure 6 represents the plot of the intensity difference versus η for different values of the squeezing parameter, and it indicates that the squeezing

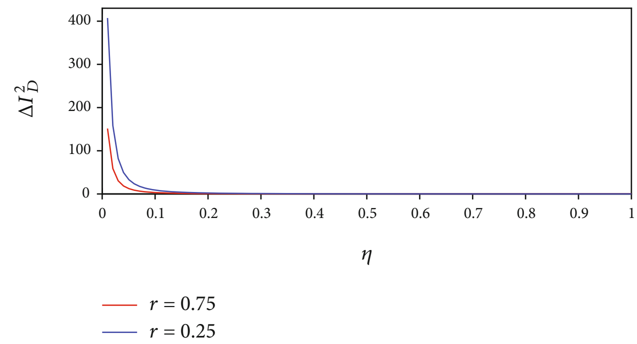

Figure 6: Plots of intensity difference fluctuation (Equation (32)) versus η for $A = 100$ and $\kappa = 0.8$ and for different values of squeeze parameter.

parameter increases as the intensity difference increases and the atomic coherence increases as the intensity decreases for $\eta > 0.19$; the $I_D = 0$ for other values of η also from Figure 7 describes the effect of the squeezing parameter on intensity difference fluctuation; furthermore, we obtain the maximum intensity difference for small values of atomic coherence at which our squeezing degree is maximum.

The variance of the intensity difference per the mean photon number of the two-mode light can be written as

$$\widehat{I}_D^2 = 1 + \frac{\langle\alpha^*(t)\alpha(t)\rangle^2 + \langle\alpha^*(t)\beta(t)\rangle^2 - 2\langle\alpha^*(t)\beta^*(t)\rangle\langle\alpha(t)\beta(t)\rangle}{\langle\alpha^*(t)\alpha(t)\rangle + \langle\beta^*(t)\beta(t)\rangle}. \tag{32}$$

The more simplified intensity difference fluctuation can be obtained by substituting Equations (21)–(24) into Equation (32).

It is not difficult to see from Figure 7 that the variance of the intensity difference increases as the squeeze parameter increases. In particular, as the variance of the intensity difference is found to be zero when η approaches 1 for all values of the squeeze parameter, since there is no possibility for emission of the photon of both modes when the atoms are initially populated in the lower level. In the same way, the variance of the intensity difference turns out to be zero when $r = 0$ for all values of η, since there is no radiation in the cavity. In relation, one can infer that the variance of the intensity difference would be relatively larger in a region where the squeezing and entanglement are significant.

4.3. Photon Number Correlation. The photon number correlation for two modes of a radiation can be defined as

$$g_{(\widehat{n}_a,\widehat{n}_b)} = \frac{\langle\widehat{n}_a\widehat{n}_b\rangle}{\langle\widehat{n}_a\rangle\langle\widehat{n}_b\rangle}, \tag{33}$$

in which

$$\begin{aligned}
\langle\widehat{n}_a\widehat{n}_b\rangle &= \left\langle a\wedge^\dagger\widehat{a}b\wedge^\dagger\widehat{b}\right\rangle,\\
\langle\widehat{n}_a\rangle &= \left\langle a\wedge^\dagger\widehat{a}\right\rangle,\\
\langle\widehat{n}_b\rangle &= \left\langle b\wedge^\dagger\widehat{b}\right\rangle,
\end{aligned} \tag{34}$$

and the operators are in the normal order. Therefore, Equation (33) can be expressed in terms of the c-number variables associated with the normal ordering as

$$g_{(\widehat{n}_a,\widehat{n}_b)} = 1 + \frac{\langle\alpha(t)\beta(t)\rangle^2}{\langle\alpha^*(t)\alpha(t)\rangle\langle\beta^*(t)\beta(t)\rangle}. \tag{35}$$

Equation (35) describes the photon number correlation $g_{(n_a,n_b)}$ of a coherently driven three-level laser with a parametric amplifier coupled to a squeezed vacuum reservoir. From Figure 8, we can see that the photon number correlation falls below 2 for $\eta = 1$ which indicates that the squeezing and entanglement vanish in the absence of the atomic coherence. Furthermore the photon number correlation increases with A for $\eta = 0.7$ and decreases for others; one can compare the effect of the linear gain coefficient with the photon number correlation, which reveals that the photon number correlation grows rapidly as the injected atomic coherence is smaller and the linear gain coefficient is large. Moreover, from Figure 9, we can understand that the photon number correlation increases with the squeezing parameter and linear gain coefficient so that the photon number correlation gets to be minimum in the region where the squeezing is maximum.

5. Entanglement of a Two-Mode Radiation

In this section, we study the entanglement of the two-mode radiation in the cavity laser in view of different inseparability criteria. A pair of particles is entangled if their state cannot be expressed as the product of the state of their separate constituents. The preparation and manipulation of these entangled state lead to a better understanding of basic quantum principles [21, 27]. Nowadays, a lot of criteria have been developed to measure, detect, and manipulate the entanglement generated by various quantum optical devices. Here, I consider the logarithm negativity and Hillery-Zubairy criteria to quantify the degree of entanglement generated by the optical system.

5.1. Logarithmic Negativity. This quantification method is the logarithmic negativity which depicts the presence of entanglement for a two-mode continuous variable based on the negativity of the partial transposition [13]; the negative partial transpose must be parallel with respect to the entanglement monotone in order to obtain the degree of entanglement. The logarithmic negativity for a two-mode state is defined as

$$E_N = \max\,[0, -\log_2 V]. \tag{36}$$

The logarithmic negativity is combined with a negative partial transpose in another case where V represents the smallest eigenvalue of the symplectic matrix [13].

$$V = \sqrt{\frac{\zeta - \sqrt{(\zeta^2 - 4\det\Omega)}}{2}}. \tag{37}$$

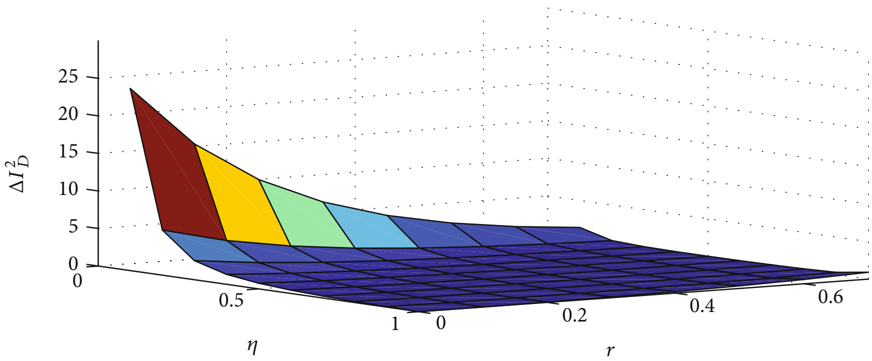

FIGURE 7: Plots of intensity difference fluctuation (Equation (32)) versus η and r for $A = 100$ and $\kappa = 0.8$.

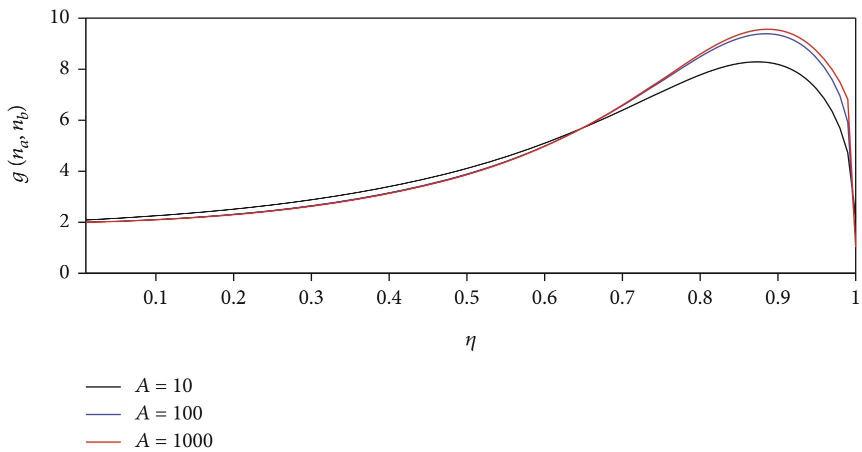

FIGURE 8: A plot of the photon number correlation versus η (Equation (35)) of the two-mode cavity radiation for $\kappa = 0.8$ and $r = 0.5$ and different values of A.

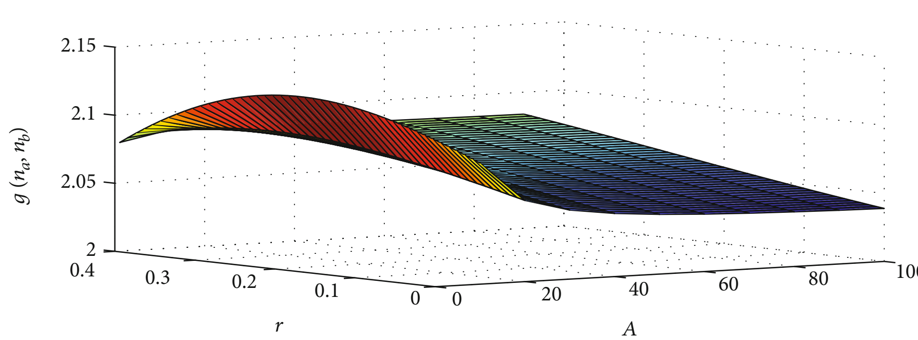

FIGURE 9: A plot of the photon number correlation of Equation (35) versus r and A of the two-mode cavity radiation for $\kappa = 0.8$ and $\eta = 0.1$.

The entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of covariance matrix $V < 1$ [13, 34].

Figure 10 clearly shows that for a smaller rate of atomic injection, the maximum degree of entanglement prefers a larger number of atoms initially prepared in the lower energy state. However, for a large rate of atomic injection, entangled light is produced when atoms are initially prepared nearly closer to the maximum atomic coherence. For instance, according to this criteria, the maximum degree of entanglement occurs at $A = 100$, and $\eta = 0.05$ is 90.7%.

The entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of covariance matrix $V < 1$, where the invariant and covariance matrices are, respectively, denoted as

$$\zeta = \det \zeta_1 + \det \zeta_2 - 2\zeta_{12}, \tag{38}$$

$$\Omega = \begin{pmatrix} \zeta_1 & \zeta_{12} \\ \zeta_{12}^T & \zeta_2 \end{pmatrix}, \tag{39}$$

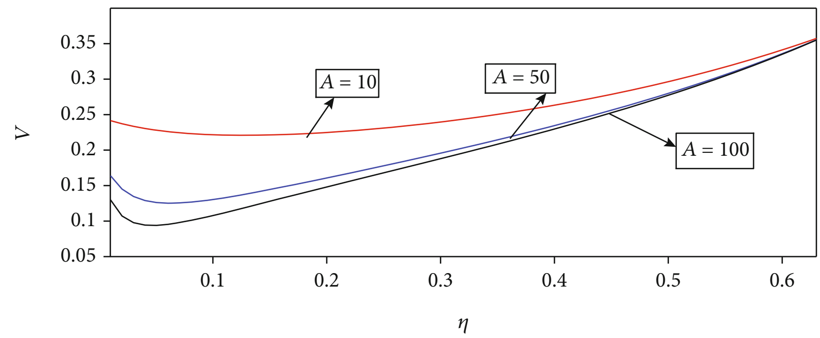

Figure 10: Plots of smallest eigenvalue V (Equation (37)) versus η for $r = 0.5$ and $\kappa = 0.8$ and for different values of linear gain coefficient.

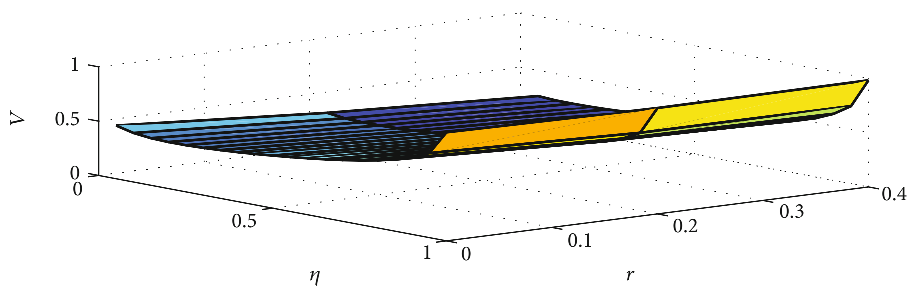

Figure 11: Plots of the smallest eigenvalue V (Equation (37)) versus η and r for $A = 100$ and $\kappa = 0.8$.

in which ζ_1 and ζ_2 are the covariance matrices describing each mode separately while ζ_{12} is the intermodal correlations.

The elements of the matrix in Equation (33) can be obtained from the relation [13]

$$\Omega_{ij} = \frac{1}{2}\left\langle \widehat{X}_i \widehat{X}_j + \widehat{X}_j \widehat{X}_i \right\rangle - \left\langle \widehat{X}_i \right\rangle \left\langle \widehat{X}_j \right\rangle, \tag{40}$$

in which $i, j = 1, 2, 3, 4$ and the quadrature operators are defined as $\widehat{X}_1 = \widehat{a} + a\wedge^{\dagger}, \widehat{X}_2 = i(a\wedge^{\dagger} - \widehat{a}), \widehat{X}_3 = \widehat{b} + b\wedge^{\dagger}$, and $\widehat{X}_4 = i(b\wedge^{\dagger} - \widehat{b})$, with this introduction of the extended covariance matrix, which can be expressed in terms of c-number variables associated with the normal ordering and noting that $m_3 = m_3^*$.

We can easily understand from Figure 11 that $V < 1$ for all values of η under consideration showing that the radiation cavity is entangled for all parameters, so it satisfies the condition predicted in the logarithm negativity, in which the logarithmic negativity for a two-mode state is defined as $E_N = \max\,[0, -\log_2 V]$; the entanglement is achieved when E_N is positive within the region of the lowest eigenvalue of covariance matrix $V < 1$.

$$\zeta = \begin{pmatrix} 2m_1 + 1 & 0 & 2m_2 & 0 \\ 0 & 2m_1 + 1 & 0 & -2m_3 \\ 2m_3 & 0 & 2m_2 + 1 & 0 \\ 0 & -2m_3 & 0 & 2m_2 + 1 \end{pmatrix}, \tag{41}$$

where $m_1 = \langle \alpha^* \alpha \rangle$, $m_2 = \langle \beta^* \beta \rangle$, and $m_3 = \langle \alpha\beta \rangle$, and its simplified form is given by Equations (22)–(24), respectively. Next, on account of Equation (39) along with the definitions of Equation (41), one can readily show that

$$\det \zeta_1 = \left(2\langle m_1 \rangle + 1\right)^2, \tag{42}$$

$$\det \zeta_2 = \left(2\langle m_2 \rangle + 1\right)^2, \tag{43}$$

$$\det \zeta_{12} = \det \zeta_{12}^T = -4\left(\langle m_3 \rangle\right)^2. \tag{44}$$

It is also possible to establish that

$$\det \Omega = \left[4\left(m_1 m_2 - m_3^2\right) + 2\left(m_1 + m_2 + 1\right)\right]^2. \tag{45}$$

The result presented in Equation (37) along with Equations (43)–(45) represents the steady-state expression of the smallest eigenvalue of the covariance matrix V for the quantum optical system.

5.2. Hillery-Zubairy (HZ) Criterion. According to the criterion introduced by Hillery-Zubairy, for two modes of the electromagnetic field with $\widehat{a}$ and $\widehat{b}$ annihilation operators, the composite state is said to be entangled if condition

$$\sqrt{\langle n_a \rangle \langle n_b \rangle} < \langle \widehat{a}\widehat{b} \rangle, \tag{46}$$

is satisfied [16], where $\widehat{n}_a$ and $\widehat{n}_b$ are the photon number operators corresponding to the involved cavity mode, whereas $\langle \widehat{a}\widehat{b} \rangle$ is the correlation of the cavity modes.

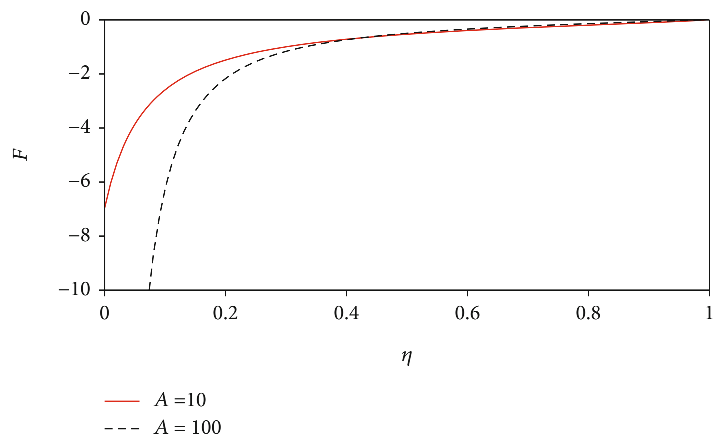

Figure 12: Plots of Equation (40) versus η for $r = 0.5$ and $\kappa = 0.8$ and for different values of linear gain coefficient.

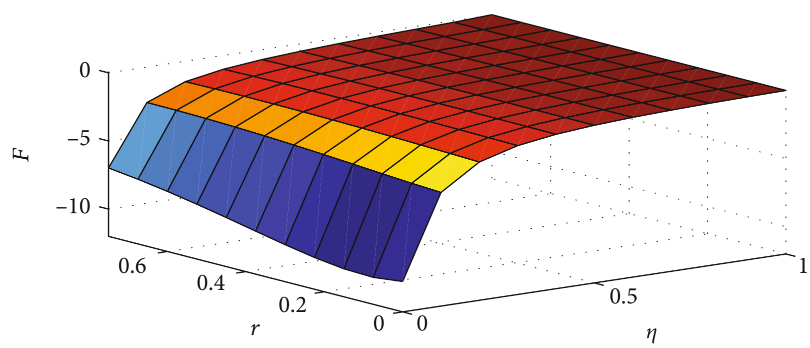

Figure 13: Plots of Equation (40) versus η and r, for $A = 100$ and $\kappa = 0.8$.

The criteria can be rewritten as

$$F = \langle n_a \rangle \langle n_b \rangle - \langle a \wedge b \rangle^2, \quad (47)$$

in which the negativity of the parameter F is a clear indication of the existence of entanglement [35]. In terms of the c-number and zero mean Gaussian variables, we see that

$$F = \langle \alpha^* \alpha \rangle \langle \beta^* \beta \rangle - \langle a \wedge b \rangle^2. \quad (48)$$

As can be seen in Figure 12, the parameter F is less than zero and becomes more negative by increasing A. In particular, F is significantly increased for $\eta < 0.3$ by decreasing A from 100 to 10. This indicates that the role of the linear gain coefficient is more pronounced at the maximum injected atomic coherence. However, we observe in Figure 13 that F closes to zero as the parameter r is increased indicating that the entanglement depletes with r. This is contrary to the other criterion and the result reported in [36] in which the driven atomic coherence is used.

6. Conclusion

In this paper, we have studied the entanglement, squeezing properties, and photon statistics of the two-mode light generated by a nondegenerate three-level laser coupled to a squeezed vacuum reservoir. First, we determined the master equation in the good-cavity limit, linear, and adiabatic approximation schemes. Applying the resulting master equation, we have derived equations of evolution of the cavity mode variables. With the aid of these equations, the quadrature variance, EPR variables, the mean number of photon, intensity difference fluctuation, and photon number correlation are obtained. We have also analyzed the squeezing and entanglement of the two-mode cavity light, and it is found that the squeezing parameter of the squeezed vacuum reservoir enhanced the degree of squeezing and entanglement. We have also seen that the degree of squeezing increases with the linear gain coefficient for small values of η, almost perfect squeezing can be obtained for large values of the linear gain coefficient, the mean photon number increases considerably due to the squeezed vacuum reservoir, and the squeezing parameter could enhance or suppress the mean number of photons based on the values of other parameters we choose. Although the degree of mean number of photons increases with squeezing parameter under various conditions, it turns out that the degree of the mean number of photons decreases with squeezing parameter when η is close to 0. Since the effect of the squeezed vacuum reservoir on the three-level laser enhanced the degree of the mean photon number, a bright and highly squeezed light was produced by the quantum optical system. Furthermore, the intermodal correlation of the infinitely many reservoir submodes leads to stronger squeezed and entangled light specially when atoms are initially prepared equally at the top and bottom levels. The maximum achievable degree of entanglement of each case increases with minimum atomic coherence. As a result, further increment of the squeezing parameter leads to the maximum degree of entanglement and squeezing to be at the maximum atomic coherence when the rate of atomic injection into the laser cavity is relatively large. Although the degree of entanglement and squeezing at the maximum atomic coherence is enhanced, the observed situation immediately reverses at the minimum atomic coherence. In other words, the intermodal correlation between submodes of the squeezed vacuum reservoir comes into play mostly when a large number of photons are available in the laser cavity. Contrary to this, the squeezing and entanglement properties of the cavity radiation decay fast at the minimum atomic coherence even if the system is coupled with the squeezed vacuum. Moreover, I showed that the entanglement quantification criteria studied by the logarithm negativity resulted in a maximum degree of entanglement of 90.3% in the presence of the squeezed vacuum reservoir; similarly, the Hillery-Zubairy criteria demonstrated the entanglement property of the cavity radiation like the logarithm negativity. However, in Hillery-Zubairy criteria, there is no lower limit on the value of the parameter F; as a result, we could not exactly know the degree of entanglement generated in this criteria, but we can clearly identify the weaker and stronger entangled light; that means, according to this criteria, the strong entangled light occurred at a more negative value of the parameter F. In general, we conclude after detailed calculations and analysis that the proposed quantum system can be utilized as a source of squeezing, entanglement, and other nonclassical and statistical features which have potential applications in different fields including modern physics, especially in quantum technologies and applications. The idea presented here is also good and may be useful for most users in various research fields, especially in quantum optics, cavity-QED, and quantum information. Moreover, the amount of entanglement and squeezing exhibited is quite robust against decoherence and hence can be used in quantum processing tasks. The introduced parameters have enhanced the squeezing, entanglement, and intensity of the cavity light.

Data Availability

The data is included in the manuscript.

Conflicts of Interest

The author declares that there is no conflict of interest regarding the publication of this paper.

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