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The Orlicz Inequality for Series of Multilinear Forms

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

The Orlicz (ℓ_2, ℓ_1)-mixed inequality of integers and fractional dimensions who states that, with a bit of extend,

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\sum_{L}\left|A_L\left(e_{j_1}, e_{j_2}\right)\right|\right)^2\right)^{\frac{1}{2}} \leq \sqrt{2}\sum_{L}\|A_L\|$$

for all sequences of bilinear forms $A_L: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$ and all positive integers n, where $\mathbb{K}^n$ denotes $\mathbb{R}^n$ or $\mathbb{C}^n$ endowed with the supremum norm. For that we follow D. Núñez-Alarcón, D. Pellegrino, and D. Serrano-Rodríguez [1] to extend this inequality to series of multilinear forms, with $\mathbb{K}^n$ endowed with $\ell_{1+\epsilon}$ norms for all successive gradually of the general $0 \leq \epsilon \leq \infty$.

Keywords: *Orlicz inequality; multilinear forms; hölder inequality; hardy-littlewood inequalities; maurey-pisier factorization.*

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1 Introduction

The origins of the theory of summability of multilinear forms and absolutely summing multilinear operators are probably associated to Orlicz (ℓ_2, ℓ_1)-mixed inequality published in the 1930's (see [2, page 24]). It states that, with a little change

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\sum_{L}\left|A_L\left(e_{j_1}, e_{j_2}\right)\right|\right)^2\right)^{\frac{1}{2}} \leq \sqrt{2}\sum_{L}\|A_L\|$$

for all bilinear forms $A_L: \mathbb{K}^n \times \mathbb{K}^n \to \mathbb{K}$, and all positive integers n. Here $\mathbb{K} = \mathbb{R}$ or $\mathbb{C}$ and $\mathbb{K}^n$ is endowed with the supremum norm. We also represent by e_k the canonical vectors in a sequence space and

$$\|A_L\| \coloneqq \sup\{|A_L(x, y)|: \|x\| \leq 1 \text{ and } \|y\| \leq 1\}.$$

An equivalent formulation is the following:

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1}, e_{j_2}\right)\right|\right)^2\right)^{\frac{1}{2}} \leq \sqrt{2}\sum_{L}\|A_L\| \tag{1}$$

for all continuous sequences of bilinear forms $A_L: c_0 \times c_0 \to \mathbb{K}$. The exponents in (1) are optimal in the sense that, fixing the exponent 1, the exponent 2 cannot be replaced by smaller exponents (nor the exponent 1 can be replaced by smaller exponents) keeping the constant independent of n. The Orlicz inequality is closely related to Littlewood's (ℓ_1, ℓ_2)-mixed inequality (see [2, page 23]), which asserts that

$$\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1}, e_{j_2}\right)\right|^2\right)^{\frac{1}{2}} \leq \sqrt{2}\sum_{L}\|A_L\|$$

for all continuous sequences of bilinear forms $A_L: c_0 \times c_0 \to \mathbb{K}$. Again, the exponents are optimal in the same sense as described above. Combining these two inequalities, and using the Hölder inequality for mixed sums we recover Littlewood's $4/3$ inequality:

$$\left(\sum_{j_1, j_2=1}^{\infty}\sum_{L}\left|A_L\left(e_{j_1}, e_{j_2}\right)\right|^{\frac{4}{3}}\right)^{\frac{3}{4}} \leq \sqrt{2}\sum_{L}\|A_L\|$$

for all continuous bilinear forms $A_L: c_0 \times c_0 \to \mathbb{K}$. For recent results on absolutely summing linear and multilinear operators see [3,4,5].

The exponent $4/3$ from the previous inequality cannot be replaced by smaller exponents keeping the constant independent of n. The constant $\sqrt{2}$ is optimal (in all the three inequalities) when $\mathbb{K} = \mathbb{R}$, but the optimal constants when $\mathbb{K} = \mathbb{C}$ are unknown.

In 1934 Hardy and Littlewood [6] (see also [7]) pushed the subject further, extending the above results to bilinear forms defined on $\ell_{1+\epsilon}$ spaces (when $\epsilon = \infty$ we consider c_0 instead of ℓ_∞). The investigation of extensions of the Hardy-Littlewood inequalities to multilinear forms were initiated by Praciano-Pereira [8] in 1981 and intensively investigated since then (see, for instance, [9-13,7,14,15]), but there are still several open problems regarding the optimal exponents and optimal constants involved.

Daniel M. Pellegrino, Anselmo Raposo Jr., and Diana M. Serrano-Rodríguez [16] explore a regularity technique to obtain optimal parameters for several results in this frame work extending generalizing theorem of Paulino [14] and others.

We shall use the same notation from [9]:

$$X_{1+\epsilon} := \begin{cases} \ell_{1+\epsilon}, \text{ if } 0 \leq \epsilon < \infty \\ c_0, \text{ if } \epsilon = \infty \end{cases}$$

and, when $\epsilon = \infty$, the sum $\left(\sum_j \|x_j\|^{1+\epsilon/\epsilon}\right)^{1/\frac{1+\epsilon}{\epsilon}}$ shall represent the supremum of $\|x_j\|$. We also denote the conjugate index of $(1+\epsilon)$ by $(1+\epsilon)^*$, i.e., $1/(1+\epsilon) + 1/(1+\epsilon)^* = 1$. We find the optimal values of the exponents $(1+\epsilon)_1, \dots, (1+\epsilon)_m$ and of the constants $(1+\epsilon)^{(\mathbb{K})(1+\epsilon)_1,\dots,(1+\epsilon)_m}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}$ satisfying

$$\left(\sum_{j_1=1}^{\infty} \cdots \left(\sum_{j_{m-1}=1}^{\infty} \left(\sum_{j_m=1}^{\infty} \sum_L \left|A_L\left(e_{j_1}, \dots e_{j_m}\right)\right|^{(1+\epsilon)_m}\right)^{\frac{(1+\epsilon)_{m-1}}{(1+\epsilon)_m}} \cdots\right)^{\frac{(1+\epsilon)_{m-2}}{(1+\epsilon)_{m-1}}}\right)^{\frac{1}{(1+\epsilon)_1}}$$

$$\leq (1+\epsilon)^{(\mathbb{K})(1+\epsilon)_1,\dots,(1+\epsilon)_m}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous m-linear forms $A_L : X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$. The answer is known in several cases (see [9,11,17] and the references therein), but a complete solution is still unknown. By [1] we shall be interested in investigating the optimal exponents $(1+\epsilon)_1, \dots, (1+\epsilon)_m$. It is simple to prove that the optimal exponent $(1+\epsilon)_m$ associated to the sum $\sum_{j_m=1}^{\infty}$ is $(1+\epsilon)_m^*$. The main result provides the optimal exponents $(1+\epsilon1,\dots,1+\epsilon m-1$ in the case that $1+\epsilon m=1+\epsilon m*$.

From now on, let $\epsilon \geq 0$, and let $(1+\epsilon)_1, \dots, (1+\epsilon)_m \in [1, \infty]$. We define $\delta^{(1+\epsilon)_k,\dots,(1+\epsilon)_m}$ and $\lambda_{2+\epsilon}^{(1+\epsilon)_k,\dots,(1+\epsilon)_m}$ by

$$\delta^{(1+\epsilon)_k,\dots,(1+\epsilon)_m} := \frac{1}{\max\left\{1 - \left(\frac{1}{(1+\epsilon)_k} + \cdots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

and

$$\lambda_{2+\epsilon}^{(1+\epsilon)_k,\dots,(1+\epsilon)_m} := \frac{1}{\max\left\{\frac{1}{2+\epsilon} - \left(\frac{1}{(1+\epsilon)_k} + \cdots + \frac{1}{(1+\epsilon)_m}\right), 0\right\}},$$

for all positive integers m and $k = 1, \dots, m$. Note that when $1/(1+\epsilon)_k + \cdots + 1/(1+\epsilon)_m \geq 1$ we have

$$\delta^{(1+\epsilon)_k,\dots,(1+\epsilon)_m} = \infty$$

and, also, when $1/(1+\epsilon)_k + \cdots + 1/(1+\epsilon)_m \geq \frac{1}{2+\epsilon}$ we have

$$\lambda_{2+\epsilon}^{(1+\epsilon)_k,\dots,(1+\epsilon)_m} = \infty.$$

The main result is, a generalization of the the Orlicz inequality. We consider the very particular case $(m, (1+\epsilon1, 1+\epsilon2 = (2, \infty, \infty)$ and σ as the identity map in its statement, we recover the Orlicz inequality (see [1]):

Theorem 1.1. Let $\epsilon \geq 0$ be an integer and $\sigma: \{1, \dots, m\} \to \{1, \dots, m\}$ be a bijection. If

$$\begin{aligned} \left(\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}\right) &\in (0, \infty]^{m-1} \\ ((1+\epsilon)_1, \dots, (1+\epsilon)_m) &\in [1, \infty]^m \end{aligned}$$

the following assertions are equivalent: (1) There is a constant $(1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_{\sigma(1)}=1}^{\infty}\left(\sum_{j_{\sigma(2)}=1}^{\infty}\cdots\left(\sum_{j_{\sigma(m)}=1}^{\infty}\sum_{L}\sum\left|A_L\left(e_{j_{\sigma(1)}}, \dots, e_{j_{\sigma(m)}}\right)\right|^{(1+\epsilon)^*_{\sigma(m)}}\right)^{\frac{(1+\epsilon/\epsilon)_{m-1}}{(1+\epsilon)^*_{\sigma(m)}}}\cdots\right)^{\frac{(1+\epsilon/\epsilon)_1}{(1+\epsilon/\epsilon)_2}}\right)^{\frac{1}{(1+\epsilon/\epsilon)_1}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1, \dots, (1+\epsilon)_m}\sum_L \|A_L\|$$

for all continuous m-linear forms $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$.

(2) The exponents $(1+\epsilon/\epsilon)_1, \dots, (1+\epsilon/\epsilon)_{m-1}$ satisfy

$$(1+\epsilon/\epsilon)_1 \geq \delta^{(1+\epsilon)_{\sigma(1)}, \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}, (1+\epsilon/\epsilon)_2 \geq \delta^{(1+\epsilon)_{\sigma(2)}, \dots, (1+\epsilon)_{\sigma(m-1)}, \mu}, \dots, (1+\epsilon/\epsilon)_{m-1} \geq \delta^{(1+\epsilon)_{\sigma(m-1)}, \mu},$$

where $\mu = \min\{(1+\epsilon)_{\sigma(m)}, 2\}$.

2 Preliminary Results

Let $0 < \epsilon < \infty$. Recall that a Banach space X has cotype $(2+\epsilon)$ if there is a constant $\epsilon > 0$ such that, we select finitely many vectors $x_1, \dots, x_n \in X$,

$$\left(\sum_{j=1}^{n}\|x_j\|^{(2+\epsilon)}\right)^{\frac{1}{(2+\epsilon)}} \leq (1+\epsilon)\left(\int_{[0,1]}\left\|\sum_{j=1}^{n}(2+\epsilon)_j(t)x_j\right\|^2 dt\right)^{\frac{1}{2}} \tag{2}$$

where $(2+\epsilon)_j$ denotes the j-th Rademacher function. The infimum of the cotypes of X is denoted by cot X.

The following result was proved in [11] (see [1]):

Theorem 2.1. (see [11]) Let $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0, \infty)^m$, and Y be an infinite-dimensional Banach space with cotype $\cot\,Y$. If

$$\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_m} < \frac{1}{\cot Y}, \tag{3}$$

then the following assertions are equivalent:

(a) There is a constant $(1+\epsilon)^Y_{(1+\epsilon)_1, \dots, (1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_{l} \|A_L\|$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$.

(b) The exponents $(2+\epsilon)_1, \dots, (2+\epsilon)_m$ satisfy

$$(2+\epsilon)_1 \geq \lambda_{\cot Y}^{(1+\epsilon)_1,\dots,(1+\epsilon)_m}, (2+\epsilon)_2 \geq \lambda_{\cot Y}^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1}$$

$$\geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

We need the following extension of the previous theorem, relaxing the hypothesis (3). Besides, below we have $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty]^m$ while in Theorem 2.1 we have $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty)^m$ (see [1]).

Theorem 2.2. Let $((2+\epsilon)_1, \dots, (2+\epsilon)_m) \in (0,\infty]^m, ((1+\epsilon)_1, \dots, (1+\epsilon)_m) \in [1,\infty]^m$ and Y be an infinite-dimensional Banach space with cotype $\cot Y$. The following assertions are equivalent:

(a) There is a constant $(1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\|A_L(e_{j_1},\dots,e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(1+\epsilon/\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_{L} \|A_L\| \tag{4}$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$.

(b) The exponents $(2+\epsilon)_1, \dots, (2+\epsilon)_m$ satisfy

$$(2+\epsilon)_1 \geq \lambda_{\cot Y}^{(1+\epsilon)_1,\dots,(1+\epsilon)_m}, (2+\epsilon)_2 \geq \lambda_{\cot Y}^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1}$$

$$\geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

Proof. We begin by proving the direct implication. We just need to consider the case

$$\frac{1}{(1+\epsilon)_1} + \cdots + \frac{1}{(1+\epsilon)_m} \geq \frac{1}{\cot Y}, \tag{5}$$

since the other case is covered by Theorem 2.1. By the Maurey-Pisier factorization result (see [18, pages 286,287]), the Banach space Y finitely factors the formal inclusion $\ell_{\cot Y} \hookrightarrow \ell_\infty$, i.e., there are universal constants $\epsilon > 0$ such that, for all n, there are vectors $z_1^L, \dots, z_n^L \in Y$ satisfying

$$(1+\epsilon)\left\|(a_j)_{j=1}^{n}\right\|_{\infty} \leq \left\|\sum_{j=1}^{n}\sum_{L} a_j z_j^L\right\| \leq (1+2\epsilon)\left(\sum_{j=1}^{n}|a_j|^{\cot Y}\right)^{\frac{1}{\cot Y}}, \tag{6}$$

for all sequences of scalars $(a_j)_{j=1}^{n}$. Consider the continuous m-linear operator $(A_L)_n: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$ given by

$$(A_L)_n\big(x^{(1)},\dots,x^{(m)}\big)=\sum_{j=1}^{n}\sum_{L}x_j^{(1)}x_j^{(2)}\cdots x_j^{(m)}z_j^L. \tag{7}$$

By (6) and the Hölder inequality we have

$$\|(A_L)_n\|=\sup_{\|x^{(1)}\|_{(1+\epsilon)_1}\le 1,\dots,\|x^{(m)}\|_{(1+\epsilon)_m}\le 1}\left\|\sum_{j=1}^{n}\sum_{L}x_j^{(1)}\dots x_j^{(m)}z_j^L\right\| \tag{8}$$

$$\le \sup_{\|x^{(1)}\|_{(1+\epsilon)_1}\le 1,\dots,\|x^{(m)}\|_{(1+\epsilon)_m}\le 1}(1+2\epsilon)\left(\sum_{j=1}^{n}\left|x_j^{(1)}\dots x_j^{(m)}\right|^{\cot Y}\right)^{1/\cot Y}$$

$$\le \sup_{\|x^{(1)}\|_{(1+\epsilon)_1}\le 1,\dots,\|x^{(m)}\|_{(1+\epsilon)_m}\le 1}(1+2\epsilon)\left(\prod_{k=1}^{m}\left(\sum_{j=1}^{n}\left|x_j^{(k)}\right|^{(1+\epsilon)_k}\right)^{1/(1+\epsilon)_k}\right)$$

$$=(1+2\epsilon).$$

Note that, by (7), we have

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\sum_{L}\|(A_L)_n(e_{j_1},\dots,e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$=\left(\sum_{j=1}^{n}\sum_{L}\|z_j^L\|^{(2+\epsilon)_1}\right)^{\frac{1}{(2+\epsilon)_1}}.$$

Thus, by (6) we conclude that

$$\left(\sum_{j_1=1}^{n}\left(\sum_{j_2=1}^{n}\cdots\left(\sum_{j_m=1}^{n}\sum_{L}\|(A_L)_n(e_{j_1},\dots,e_{j_m})\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_1}{(2+\epsilon)_2}}\right)^{\frac{1}{(2+\epsilon)_1}}$$

$$\ge (1+\epsilon)n^{\frac{1}{(2+\epsilon)_1}}$$

Combining the previous inequality with (4) and (8) we conclude that

$$(1+\epsilon)n^{1/(2+\epsilon)_1}\le (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}(1+2\epsilon).$$

Thus, since n is arbitrary, we have

$$(2+\epsilon)_1=\infty=\lambda_{\cot Y}^{(1+\epsilon)_1,\dots,(1+\epsilon)_m}. \tag{9}$$

If

$$\frac{1}{(1+\epsilon)_i}+\cdots+\frac{1}{(1+\epsilon)_m}\geq\frac{1}{\cot\,Y}$$

for all i, the proof is immediate. Otherwise, let $i_0\in\{2,3,\dots,m\}$ be the smallest index such that

$$\begin{cases}\dfrac{1}{(1+\epsilon)_{i_0}}+\cdots+\dfrac{1}{(1+\epsilon)_m}<\dfrac{1}{\cot\,Y},\\ \dfrac{1}{(1+\epsilon)_{i_0-1}}+\cdots+\dfrac{1}{(1+\epsilon)_m}\geq\dfrac{1}{\cot\,Y}.\end{cases}$$

If $i_0=2$, note that by (9) we have

$$\sup_{j_1}\left(\sum_{j_2=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\dots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}}\right)^{\frac{1}{(2+\epsilon)_2}}$$

$$\leq(1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_L\|A_L\| \qquad (10)$$

for all continuous m-linear operators $A_L\colon X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. From (10) it is simple to show that

$$\left(\sum_{j_2=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_2},\dots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_2}{(2+\epsilon)_3}}\right)^{\frac{1}{(2+\epsilon)_2}}$$

$$\leq(1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_L\|A_L\|,$$

for all continuous $(m-1)$-linear operators $A_L\colon X_{(1+\epsilon)_2}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Since

$$\frac{1}{(1+\epsilon)_2}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot Y},$$

by Theorem 2.1 we conclude that

$$(2+\epsilon)_2\geq\lambda^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}_{\cot\,Y},(2+\epsilon)_3\geq\lambda^{(1+\epsilon)_3,\dots,(1+\epsilon)_m}_{\cot\,Y},\dots,(2+\epsilon)_{m-1}$$
$$\geq\lambda^{(1+\epsilon)_{m-1},(1+\epsilon)_m}_{\cot Y},(2+\epsilon)_m\geq\lambda^{(1+\epsilon)_m}_{\cot Y}.$$

If $i_0=3$, we consider

$$A_L\left(x^{(1)},\dots,x^{(m)}\right)=x_1^{(1)}\sum_{j=1}^{n}\sum_L x_j^{(2)}\cdots x_j^{(m)}z_j^L$$

and we can imitate the previous arguments to conclude that

$$(2+\epsilon)_2 = \infty = \lambda_{\cot Y}^{(1+\epsilon)_2,\dots,(1+\epsilon)_m}.$$

and hence

$$\sup_{j_1,j_2}\left(\sum_{j_3=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_1},\dots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_3}{(2+\epsilon)_4}}\right)^{\frac{1}{(2+\epsilon)_3}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}^{Y}\sum_{L}\|A_L\|, \tag{11}$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Again, it is plain that

$$\left(\sum_{j_3=1}^{\infty}\left(\cdots\left(\sum_{j_m=1}^{\infty}\sum_{L}\left\|A_L\left(e_{j_3},\dots,e_{j_m}\right)\right\|^{(2+\epsilon)_m}\right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}}\cdots\right)^{\frac{(2+\epsilon)_{i_0+1}}{(2+\epsilon)_{i_0}}}\right)^{\frac{1}{(2+\epsilon)_{i_0}}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}^{Y}\sum_{L}\|A_L\|$$

for all continuous $(m-2)$-linear operators $A_L: X_{(1+\epsilon)_3}\times\cdots\times X_{(1+\epsilon)_m}\to Y$. Since

$$\frac{1}{(1+\epsilon)_3}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot Y},$$

by Theorem 2.1 we have

$$(2+\epsilon)_3 \geq \lambda_{\cot Y}^{(1+\epsilon)_3,\dots,(1+\epsilon)_m}, (2+\epsilon)_4 \geq \lambda_{\cot Y}^{(1+\epsilon)_4,\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_{m-1}$$

$$\geq \lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

We conclude the proof in a similar fashion for $i_0 = 4,\dots,m$.

Now we prove the reverse implication. The case

$$\frac{1}{(1+\epsilon)_1}+\cdots+\frac{1}{(1+\epsilon)_m}<\frac{1}{\cot Y},$$

is encompassed by Theorem 2.1. So, we shall consider

$$\frac{1}{(1+\epsilon)_1}+\cdots+\frac{1}{(1+\epsilon)_m}\geq\frac{1}{\cot Y}.$$

If

$$\frac{1}{(1+\epsilon)_i}+\cdots+\frac{1}{(1+\epsilon)_m}\geq\frac{1}{\cot Y}$$

for all i, the proof is immediate. Otherwise, let $i_0 \in \{2, \dots, m\}$ be the smallest index such that

$$\begin{cases} \dfrac{1}{(1+\epsilon)_{i_0}} + \cdots + \dfrac{1}{(1+\epsilon)_m} < \dfrac{1}{\cot Y}, \\ \dfrac{1}{(1+\epsilon)_{i_0-1}} + \cdots + \dfrac{1}{(1+\epsilon)_m} \geq \dfrac{1}{\cot Y}. \end{cases}$$

We need to prove that there is a constant $(1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$, such that

$$\sup_{j_1,\dots,j_{i_0-1}} \left(\sum_{j_{i_0}=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}} \right)^{\frac{1}{(2+\epsilon)_{i_0}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_L \|A_L\|$$

for

$$(2+\epsilon)_{i_0} \geq \lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}$$

By Theorem 2.1, we know that for any fixed vectors $e_{j_1}, \dots, e_{j_{i_0-1}}$, there is a constant $(1+\epsilon)^{Y}_{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m} \geq 1$, such that

$$\left(\sum_{j_{i_0}=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}} \cdots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\dots,(1+\epsilon)_m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous m-linear operators $A: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$. Then,

$$\sup_{j_1,\dots,j_{i_0-1}} \left(\sum_{j_{i_0}=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_L \|A_L(e_{j_1}, \dots, e_{j_m})\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}} \cdots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\dots,(1+\epsilon)_m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}}$$

$$\leq (1+\epsilon)^{Y}_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_L \|A_L\|$$

for all continuous m-linear operators $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to Y$.

To conclude the proof we just need to remark that

$$\sup_{j_1,\dots,j_{i_0-1}} \left(\sum_{j_{i_0}=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_{L} \left\| A_L\left(e_{j_1},\dots,e_{j_m}\right) \right\|^{(2+\epsilon)_m} \right)^{\frac{(2+\epsilon)_{m-1}}{(2+\epsilon)_m}} \cdots \right)^{\frac{(2+\epsilon)_{i_0}}{(2+\epsilon)_{i_0+1}}} \right)^{\frac{1}{(2+\epsilon)_i}}$$

$$\leq \sup_{j_1,\dots,j_{i_0-1}} \left(\sum_{j_{i_0}=1}^{\infty} \left(\cdots \left(\sum_{j_m=1}^{\infty} \sum_{L} \left\| A_L\left(e_{j_1},\dots,e_{j_m}\right) \right\|^{\lambda_{\cot Y}^{(1+\epsilon)_m}} \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{m-1},(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_m}}} \cdots \right)^{\frac{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0+1},\dots,(1+\epsilon)_m}}} \right)^{\frac{1}{\lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}}}$$

provided

$$(2+\epsilon)_{i_0} \geq \lambda_{\cot Y}^{(1+\epsilon)_{i_0},\dots,(1+\epsilon)_m}, \dots, (2+\epsilon)_m \geq \lambda_{\cot Y}^{(1+\epsilon)_m}.$$

3 Proof of Theorem 1.1 (See [1])

Let the adjoint of a Banach space X be denoted by X^*. To simplify the notation we will consider $\sigma(j) = j$ for all j; the other cases are similar. Let $\mathcal{L}^m\left(X_{(1+\epsilon)_1},\dots,X_{(1+\epsilon)_m};Y\right)$ denote the space of all continuous m-linear operators from $X_{p_1} \times \cdots \times X_{p_m}$ to Y. By the canonical isometric isomorphism

$$\Psi_L: \mathcal{L}^m\left(X_{(1+\epsilon)_1},,X_{(1+\epsilon)_m};\mathbb{K}\right) \to \mathcal{L}^{m-1}\left(X_{(1+\epsilon)_1},\dots,X_{(1+\epsilon)_{m-1}};\left(X_{(1+\epsilon)_m}\right)^*\right)$$

and duality in $X_{(1+\epsilon)_m}$, note that, if $R \in \mathcal{L}^m\left(X_{(1+\epsilon)_1},\dots,X_{(1+\epsilon)_m};\mathbb{K}\right)$, we have

$$R(x_1,\dots,x_{m-1},e_n) = \Psi_L(R)(x_1,\dots,x_{m-1})(e_n) = \left(\Psi_L(R)(x_1,\dots,x_{m-1})\right)_n. \tag{12}$$

We start off by proving (1)⇒(2). Let us suppose that there is a constant $(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \geq 1$ such that

$$\left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_m=1}^{\infty} \sum_{L} \left| T_L\left(e_{j_1},\dots,e_{j_m}\right) \right|^{(1+\epsilon)_m^*} \right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}} \cdots \right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}} \right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m} \sum_{L} \|T_L\| \tag{13}$$

for all continuous m-linear forms $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$

Consider a continuous $(m-1)$-linear operator $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \to \left(X_{(1+\epsilon)_m}\right)^*$. Then, using our hypothesis, we have

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\sum_{L}\|A_L(e_{j_1},\dots,e_{j_{m-1}})\|_{(X_{(1+\epsilon)_m})^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}} \tag{14}$$

$$= \left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_{L}\left|\left(A_L(e_{j_1},\dots,e_{j_{m-1}})\right)_{j_m}\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$\overset{(12)}{=} \sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_{L}\left|\Psi_L^{-1}(A_L)(e_{j_1},\dots,e_{j_m})\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\Bigg)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$\le (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_{L}\|\Psi_L^{-1}(A_L)\|$$

$$\le (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_m}\sum_{L}\|A_L\|$$

for all continuous $(m-1)$-linear operators $A_L\colon X_{(1+\epsilon)_1}\times\cdots\times X_{(1+\epsilon)_{m-1}}\to\left(X_{(1+\epsilon)_m}\right)^*$. Since $\left(X_{(1+\epsilon)_m}\right)^*$ has cotype $\max\{(1+\epsilon)_m^*,2\}$, by Theorem 2.2, the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1,\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ in (2.2) satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \ge \lambda_{\max\{(1+\epsilon)_m^*,2\}}^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \ge \lambda_{\max\{(1+\epsilon)_m^*,2\}}^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1}},\dots,\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$$

$$\ge \lambda_{\max\{(1+\epsilon)_m^*,2\}}^{(1+\epsilon)_{m-1}}. \tag{15}$$

Since

$$1-\frac{1}{\max\{(1+\epsilon)_m^*,2\}}=\frac{1}{\mu}$$

we have

$$\lambda_{\max\{(1+\epsilon)_m^*,2\}}^{(1+\epsilon)_i,\dots,(1+\epsilon)_{m-1}} = \frac{1}{\max\left\{\frac{1}{\max\{(1+\epsilon)_m^*,2\}} - \left(\frac{1}{(1+\epsilon)_i} + \cdots + \frac{1}{(1+\epsilon)_{m-1}}\right), 0\right\}}$$
$$= \frac{1}{\max\left\{1 - \left(\frac{1}{(1+\epsilon)_i} + \cdots + \frac{1}{(1+\epsilon)_{m-1}} + \frac{1}{\mu}\right), 0\right\}}$$
$$= \delta^{(1+\epsilon)_i,\dots,(1+\epsilon)_{m-1},\mu}$$

for all $i \in \{1, \dots, m-1\}$. Then, (15) can be re-stated as

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \delta^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1},\mu}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \delta^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1},\mu}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \delta^{(1+\epsilon)_{m-1},\mu}$$

and the proof is done.

(2)⇒(1). If the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \delta^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1},\mu}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \delta^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1},\mu}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \delta^{(1+\epsilon)_{m-1},\mu},$$

we have, again, that the exponents $\left(\frac{1+\epsilon}{\epsilon}\right)_1, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}$ satisfy

$$\left(\frac{1+\epsilon}{\epsilon}\right)_1 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}, \left(\frac{1+\epsilon}{\epsilon}\right)_2 \geq \lambda_{2+\epsilon}^{(1+\epsilon)_2,\dots,(1+\epsilon)_{m-1}}, \dots, \left(\frac{1+\epsilon}{\epsilon}\right)_{m-1} \geq \lambda_{2+\epsilon}^{(1+\epsilon)_{m-1}},$$

with $(2+\epsilon) = \cot\left(X_{(1+\epsilon)_m}\right)^*$. Thus, by Theorem 2.2, there is a constant

$$(1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*} \geq 1$$

such that

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\sum_L \|T_L(e_{j_1},\dots,e_{j_{m-1}})\|_{\left(X_{(1+\epsilon)_m}\right)^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$
$$\leq (1+\epsilon)_{(1+\epsilon)_1,\dots,(1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*}\sum_L \|T_L\|$$

for all continuous m-linear operators $T_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_{m-1}} \to \left(X_{(1+\epsilon)_m}\right)^*$.

We thus have

$$\left(\sum_{j_1=1}^{\infty}\left(\sum_{j_2=1}^{\infty}\cdots\left(\sum_{j_{m-1}=1}^{\infty}\left(\sum_{j_m=1}^{\infty}\sum_L \left|A_L(e_{j_1},\dots,e_{j_m})\right|^{(1+\epsilon)_m^*}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}{(1+\epsilon)_m^*}}\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}}\cdots\right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}}\right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$= \left(\sum_{j_1=1}^{\infty} \left(\sum_{j_2=1}^{\infty} \cdots \left(\sum_{j_{m-1}=1}^{\infty} \sum_{L} \left\| \Psi_L(A_L)\left(e_{j_1}, \ldots, e_{j_m}\right) \right\|_{\left(X_{(1+\epsilon)_m}\right)^*}^{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}} \right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-2}}{\left(\frac{1+\epsilon}{\epsilon}\right)_{m-1}}} \cdots \right)^{\frac{\left(\frac{1+\epsilon}{\epsilon}\right)_1}{\left(\frac{1+\epsilon}{\epsilon}\right)_2}} \right)^{\frac{1}{\left(\frac{1+\epsilon}{\epsilon}\right)_1}}$$

$$\leq (1+\epsilon)_{(1+\epsilon)_1, \ldots, (1+\epsilon)_{m-1}}^{\left(X_{(1+\epsilon)_m}\right)^*} \sum_{L} \|\Psi_L(A_L)\|$$

$$= (1+\epsilon)_{(1+\epsilon)_1, \ldots, (1+\epsilon)_{m-1}} \sum_{L} \|A_L\|$$

for all continuous m-linear forms $A_L: X_{(1+\epsilon)_1} \times \cdots \times X_{(1+\epsilon)_m} \to \mathbb{K}$.

Remark 3.1. [1] proved that the determination of the exact values of the constants involved in the main theorem is probably a difficult task, as it happens with the Hardy-Littlewood inequalities (see [19,20] and the references therein). However when we are restricted to the bilinear case, with $(1+\epsilon)_1 = (1+\epsilon)_2 = \infty$ and σ as the identity map, it is not difficult to check that we recover the constant $\sqrt{2}$ from the Orlicz inequality.

4 Conclusion

We state theoretically the Pioneer Orlicz (ℓ_2, ℓ_1)-mixed inequality with $\ell_{1+\epsilon}$ norms for the systematically improved in the higher inequalities of $\ell^{1+\epsilon}$ space: of the discrete cases for best bounds and estimate of the exponents and optimality of less small domains.

Competing Interests

Authors have declared that no competing interests exist.

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