Asian Research Journal of Mathematics

18(6): 62-69, 2022; Article no.ARJOM.87167
ISSN: 2456-477X

Global Clustering Coefficient of the Products of Complete Graphs

Remarl Joseph M. Damalerio [a*] **and Rolito G. Eballe** [a]

[a] *Department of Mathematics, College of Arts and Sciences, Central Mindanao University, Musuan, Maramag, Bukidnon-8714, Philippines.*

Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

Article Information

DOI: 10.9734/ARJOM/2022/v18i630384

Open Peer Review History:

This journal follows the Advanced Open Peer Review policy. Identity of the Reviewers, Editor(s) and additional Reviewers, peer review comments, different versions of the manuscript, comments of the editors, etc are available here: https://www.sdiarticle5.com/review-history/87167

Short Research Article

Received: 18 March 2022
Accepted: 20 May 2022
Published: 23 May 2022

Abstract

The global clustering coefficient $Cc(G)$ of a connected graph G of order at least 3 is a metric that somehow measures how close G to being a complete graph. Its value ranges from 0 to 1. In this paper, we will show that for the tensor product $K_m \otimes K_m$ and cartesian product $K_m \boxdot K_m$, $Cc(K_m \otimes K_m)$ and $Cc(K_m \boxdot K_m)$ approach to 1 and 1/2, respectively, as $m \to \infty$.

Keywords: Clustering coefficient; tensor product; cartesian product; regular graphs.

2010 Mathematics Subject Classification: 05C09, 05C38, 05C76.

1 Introduction

Let G be a simple undirected graph with vertex set $V(G)$ and edge set $E(G)$. Let $N_G(v) = \{u \in V(G) : uv \in E(G)\}$ be the open neighborhood of a vertex $v \in V(G)$, $\deg_G v$ the degree of v, and $t_G(v) = |E(\langle N_G(v) \rangle)|$ the number of triangles in G which are incident to v. The *local clustering coefficient* of vertex v in G, denoted by $Cc_v(G)$, is a measure that evaluates the local triangle density of G at the level of vertex v. This number $Cc_v(G)$ can be defined as

**Corresponding author: E-mail: rjdamalerio@gmail.com;*

$$Cc_v(G) = \begin{cases} 0, & \text{if } \deg_G v \leq 1, \\ \frac{t_G(v)}{\binom{\deg_G v}{2}}, & \text{if } \deg_G v \geq 2. \end{cases} \tag{1.1}$$

This formula is a unifying version between its treatment in [1] and [2]. On the other hand, the *g*lobal clustering coefficient $Cc(G)$ of a graph G with order n is a measure that indicates the overall clustering of G, obtained by averaging the local clustering coefficients of all the vertices in G. That is, $Cc(G) = 0$ if $\deg_G v \leq 1$ for each $v \in V(G)$; otherwise,

$$Cc(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \geq 2}} Cc_v(G) = \frac{1}{n} \sum_{\substack{v \in V(G) \\ \deg_G v \geq 2}} \frac{2t_G(v)}{\deg_G v(\deg_G v - 1)}. \tag{1.2}$$

This measure was introduced in the field of social network analysis by Duncan J. Watts and Steven Strogatz [3] in 1998, where one of its goals was to determine whether a graph was a "small-world network". Since then, several studies from various perspectives have also emerged. In [4], the authors gave some expressions and bounds for the global clustering coefficient of the tensor product of graphs, although a related study on finding the number of distinct triangles in the tensor product $G \otimes H$ was done in [5], while a triangle-counting algorithm for large networks appeared in [6].

In this paper, we investigate the global clustering coefficients of the tensor and cartesian product of complete graphs using some properties that the tensor and cartesian product hold and some inherent characteristics possessed by the complete graphs K_m such as in the observation given below. The scope of this work falls within the general motivation of investigating graphs under some binary operations and expressing some of their parameterized values in terms of some relevant invariants of the constituent graphs such as the ones done in [7, 8, 9, 10]. Our final goal in this study is to prove that for the tensor product $K_m \otimes K_m$ and cartesian product $K_m \boxdot K_m$, $Cc(K_m \otimes K_m)$ and $Cc(K_m \boxdot K_m)$ approach to 1 and 1/2, respectively, as $m \to \infty$. For basic graph theory terminologies not specifically described nor defined in this paper, please refer to either [11] or [12].

Lemma 1.1 *For the complete graph K_m,*

$$Cc(K_m) = \begin{cases} 1 \text{ if } m \geq 3, \\ 0 \text{ if } m = 1,\ 2. \end{cases}$$

Proof: This is immediate from Equation (1.2). □

2 Tensor Product of Complete Graphs

The *tensor product* $G \otimes H$ of two graphs G and H is the graph with vertex set $V(G \otimes H) = V(G) \times V(H)$ and edge set $E(G \otimes H)$ satisfying the following adjacency condition: $(u, v)(u', v') \in E(G \otimes H)$ if and only if $uu' \in E(G)$ and $vv' \in E(H)$. A *regular graph* is a graph that has uniform degree in its vertices. If G is a regular graph with degree d in all its vertices, then we call G a d-regular graph. In [4], Damalerio and Eballe gave a formula for the global clustering coefficient of the tensor product of regular graphs in terms of the global clustering coefficient of each factor.

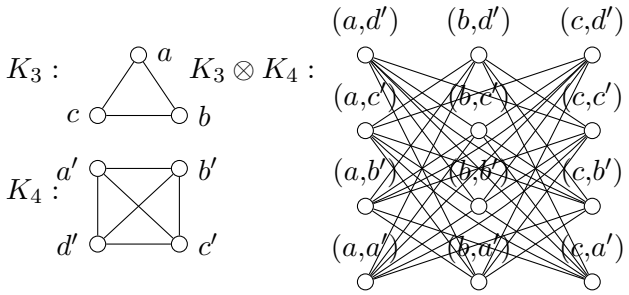

Fig. 1. The complete graphs K_3, K_4, and their tensor product $K_3 \otimes K_4$

If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then Theorem 3.1 in [4] asserts that

$$Cc(G \otimes H) = f \cdot Cc(G) \cdot Cc(H), \tag{2.1}$$

where $f = (d_G - 1)(d_H - 1)/(d_G \cdot d_H - 1)$.

Theorem 2.1 *For the tensor product $K_m \otimes K_n$, where m,$n \geq 3$,*

$$Cc(K_m \otimes K_n) = \frac{mn - 2m - 2n + 4}{mn - m - n}.$$

Proof: Note that K_m and K_n are regular graphs with regularities $m - 1$ and $n - 1$, respectively. Using Lemma 1.1 and Equation (2.1), we have

$$\begin{aligned} Cc(K_m \otimes K_n) =& \frac{(m-2)(n-2)}{(m-1)(n-1)-1} \cdot Cc(K_m) \cdot Cc(K_n) \\ =& \frac{mn - 2m - 2n + 4}{mn - m - n} \cdot 1 \cdot 1. \end{aligned}$$

□

In the next result, we give an asymptotic value to $Cc(K_m \otimes K_n)$, where $n \geq 3$ is held constant and $m \to \infty$.

Corollary 2.2 *If $n \geq 3$ is considered constant, then $Cc(K_m \otimes K_n) \to \frac{n-2}{n-1}$ as $m \to \infty$.*

Proof: Using Theorem 2.1, we have

$$\lim_{m\to\infty} Cc(K_m \otimes K_n) = \lim_{m\to\infty} \frac{n - 2 - \frac{2n}{m} + \frac{4}{m}}{n - 1 - \frac{n}{m}} = \frac{n-2}{n-1}.$$

□

Corollary 2.3 *For the tensor product $K_m \otimes K_m$, $Cc(K_m \otimes K_m) \to 1$ as $m \to \infty$.*

Proof: Using Theorem 2.1, we have

$$\lim_{m\to\infty} Cc(K_m \otimes K_m) = \lim_{m\to\infty} \frac{m^2 - 4m + 4}{m^2 - 2m} = \lim_{m\to\infty} \frac{1 - \frac{4}{m} + \frac{4}{m^2}}{1 - \frac{2}{m}} = 1.$$

□

Actually, the values of $Cc(K_m \otimes K_m) = \frac{m^2-4m+4}{m^2-2m}$ can be shown to be strictly increasing. The next result gives an asymptotic value to $Cc(K_m \otimes K_n)$, wherein both orders of the complete graphs, m and n, approach positive infinity.

Corollary 2.4 *For tensor product $K_m \otimes K_n$, $Cc(K_m \otimes K_n) \to 1$ as both m, $n \to \infty$.*

Proof: Using Theorem 2.1, we have

$$\lim_{m,n\to\infty} Cc(K_m \otimes K_n) = \lim_{m,n\to\infty} \frac{mn - 2m - 2n + 4}{mn - m - n} = \lim_{m,n\to\infty} \frac{1 - \frac{2}{n} - \frac{2}{m} + \frac{4}{mn}}{1 - \frac{1}{n} - \frac{1}{m}} = 1. \qquad \square$$

3 Cartesian Product of Complete Graphs

Recall that the *cartesian product* $G \boxdot H$ of two graphs G and H is the graph with vertex set $V(G \boxdot H) = V(G) \times V(H)$ and edge-set $E(G \boxdot H)$ satisfying the following conditions: $(u, v)(u', v') \in E(G \boxdot H)$ if and only if either $uu' \in E(G)$ and $v = v'$, or $u = u'$ and $vv' \in E(H)$.

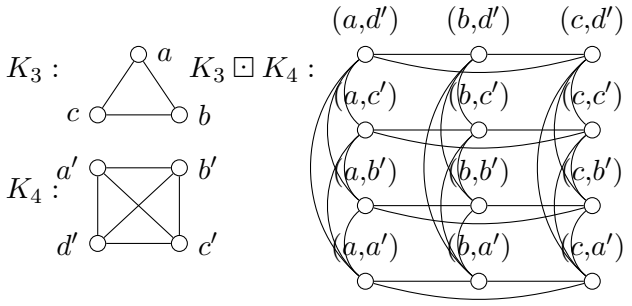

Fig. 2. The complete graphs K_3, K_4, and their cartesian product $K_3 \boxdot K_4$

Lemma 3.1 *Let G and H be graphs with orders n_1 and n_2, respectively, and sizes m_1 and m_2, respectively. If $u \in V(G)$ and $v \in V(H)$, then the following properties hold.*

1. $\deg_{G\boxdot H}(u, v) = \deg_G u + \deg_H v$,
2. $|E(G \boxdot H)| = n_1 m_2 + n_2 m_1$,
3. $t_{G\boxdot H}(u, v) = t_G(u) + t_H(v)$.

Proof: The proofs of (1) and (2) above follow directly from the definition of the cartesian product of graphs. As for (3), let $(u, v) \in V(G \boxdot H)$ such that $(u', v')(u'', v'') \in E(\langle N_{G\boxdot H}(u, v)\rangle)$. Then vertices (u, v), (u', v'), $(u'', v'') \in V(G \boxdot H)$ are pairwise adjacent in $G \boxdot H$.

Case 1: Suppose that $uu' \in E(G)$. This means that $v = v'$ is a must. Since (u, v) and (u'', v'') are adjacent in $G \boxdot H$, we must have either $u = u''$ and $vv'' \in E(H)$, or $uu'' \in E(G)$ and $v = v''$. But the option $u = u''$ and $vv'' \in E(H)$, together with the observation that $(u', v) = (u', v')$ is adjacent to $(u'', v'') = (u, v'')$ in $G \boxdot H$, leads to $u' = u$, which is imposible since we already have $uu' \in E(G)$. As a consequence, we only have $uu'' \in E(G)$ and $v = v''$. Since $v = v'$, we now have $v = v' = v''$ and, hence, $u'u'' \in E(G)$, so that $\langle\{u, u', u''\}\rangle$ is a triangle incident with vertex u in G. Thus, in this case, a particular triangle in $G \boxdot H$ incident with $(u, v) \in V(G \boxdot H)$ clearly shows a unique triangle in G incident with $u \in V(G)$.

Case 2: In this case we have $uu' \notin E(G)$ and hence we have $u = u'$ and $vv' \in E(H)$. Since $G \boxdot H \cong H \boxdot G$, we can apply the argument used in Case 1 above to a starting premise that $vv' \in E(H)$, producing a similar conclusion that in this case, a particular triangle in $G \boxdot H$ incident with $(u, v) \in V(G \boxdot H)$ would show a unique triangle in H incident with $v \in V(H)$.

The two cases together imply that every triangle in $G\boxdot H$ incident with the vertex $(u,v) \in V(G\boxdot H)$ is attributable either to a unique triangle in G incident with $u \in V(G)$ or to a unique triangle in H incident with $v \in V(H)$.

Conversely, every triangle in G(or in H) incident with $u \in V(G)$ (or with $v \in V(H)$) produces a corresponding unique triangle in $G \boxdot H$ incident with (u,y) for every $y \in V(H)$ (or with (x,v) for every $x \in V(G)$). Finally, we can now conclude that $t_{G\boxdot H}(u,v) = t_G(u) + t_H(v)$. □

Theorem 3.2 *If $u \in V(G)$ and $v \in V(H)$ such that $\deg_G u \geq 2$ and $\deg_H v \geq 2$, then the local clustering coefficient of (u,v) in $G \boxdot H$ is given by the formula*

$$Cc_{(u,v)}(G \boxdot H) = p(u,v) \cdot Cc_u(G) + q(u,v) \cdot Cc_v(H),$$

where $p(u,v) = (\deg_G u)(\deg_G u - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$ and $q(u,v) = (\deg_H v)(\deg_H v - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$.

Proof: Using Equation (1.1) and Lemma 3.1(3), we have

$$\begin{aligned} Cc_{(u,v)}(G \boxdot H) &= \frac{t_{G\boxdot H}(u,v)}{\binom{\deg_{G\boxdot H}(u,v)}{2}} = \frac{t_G(u) + t_H(v)}{\binom{\deg_G u + \deg_H v}{2}} \\ &= \frac{Cc_u(G)\binom{\deg_G u}{2} + Cc_v(H)\binom{\deg_H v}{2}}{\binom{\deg_G u + \deg_H v}{2}} \\ &= \frac{Cc_u(G)\deg_G u(\deg_G u - 1) + Cc_v(H)\deg_H v(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \\ &= \frac{Cc_u(G)\deg_G u(\deg_G u - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} + \frac{Cc_v(H)\deg_H v(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)}, \end{aligned}$$

and the claimed formula follows. □

The next result, which is for $Cc(G \boxdot H)$, is a consequence of Theorem 3.2.

Corollary 3.3 *Let G and H be graphs of orders n_1 and n_2, respectively. Suppose $\delta(G) \geq 2$ and $\delta(H) \geq 2$. Then the global clustering coefficient of $G \boxdot H$ is given by the expression*

$$Cc(GvH) = \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} p(u,v) \cdot Cc_u(G) + q(u,v) \cdot Cc_v(H),$$

where $p(u,v) = (\deg_G u)(\deg_G u - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$ and $q(u,v) = (\deg_H v)(\deg_H v - 1)/[(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)]$.

Proof: Using Equation (1.2) and Theorem 3.2, we obtain

$$\begin{aligned} Cc(G \boxdot H) =& \frac{1}{n_1 n_2} \sum_{(u,v) \in V(G\boxdot H)} Cc_{(u,v)}(G \boxdot H) \\ =& \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} \left(\frac{Cc_u(G)(\deg_G u)(\deg_G u - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \right. \\ & \left. + \frac{Cc_v(H)(\deg_H v)(\deg_H v - 1)}{(\deg_G u + \deg_H v)(\deg_G u + \deg_H v - 1)} \right). \end{aligned}$$

The claimed formula follows. □

Theorem 3.4 *Let G and H be graphs with orders n_1 and n_2, respectively. If G and H are regular graphs with regularities $d_G \geq 2$ and $d_H \geq 2$, respectively, then*

$$Cc(G \boxdot H) = p \cdot Cc(G) + q \cdot Cc(H),$$

where $p = d_G(d_G - 1)/[(d_G + d_H)(d_G + d_H - 1)]$ and $q = d_H(d_H - 1)/[(d_G + d_H)(d_G + d_H - 1)]$.

Proof: Using Equation (1.2), Corollary 3.3, and the fact that $G \boxdot H$ is also a regular graph with regularity $d_{G\boxdot H} = d_G + d_H$ from Lemma 3.1(1), we have

$$\begin{aligned}
Cc(G \boxdot H) =& \frac{1}{n_1 n_2} \sum_{u \in V(G)} \sum_{v \in V(H)} \left(\frac{Cc_u(G) d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} + \frac{Cc_v(H) d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \right) \\
=& \frac{1}{n_1 n_2} \left(\left(\frac{n_2 d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} \sum_{u \in V(G)} Cc_u(G) \right) \right. \\
& \left. + \left(\frac{n_1 d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \sum_{v \in V(H)} Cc_v(H) \right) \right) \\
=& \frac{d_G(d_G - 1)}{(d_G + d_H)(d_G + d_H - 1)} \cdot Cc(G) + \frac{d_H(d_H - 1)}{(d_G + d_H)(d_G + d_H - 1)} \cdot Cc(H),
\end{aligned}$$

and the claimed formula holds. □

Theorem 3.5 *For the cartesian product $K_m \boxdot K_n$, where m, $n \geq 3$,*

$$Cc(K_m \boxdot K_n) = \frac{m^2 - 3m + n^2 - 3n + 4}{m^2 - 5m + 2mn + n^2 - 5n + 6}.$$

Proof: Given that K_m is $(m-1)$-regular, K_n is $(n-1)$-regular, and $Cc(K_m) = Cc(K_n) = 1$, Theorem 3.4 asserts that

$$\begin{aligned}
Cc(K_m \boxdot K_n) =& \frac{(m-1)(m-2)}{(m+n-2)(m+n-3)} + \frac{(n-1)(n-2)}{(m+n-2)(m+n-3)} \\
=& \frac{(m-1)(m-1) + (n-1)(n-2)}{(m+n-2)(m+n-3)} \\
=& \frac{m^2 - 3m + n^2 - 3n + 4}{m^2 - 5m + 2mn + n^2 - 5n + 6},
\end{aligned}$$

which completes the proof. □

The next result gives an asymptotic value to $Cc(K_m \boxdot K_m)$ as $m \to \infty$.

Corollary 3.6 *For the cartesian product $K_m \boxdot K_m$, $Cc(K_m \boxdot K_m) \to \frac{1}{2}$ as $m \to \infty$.*

Proof: Using Corollary 3.5, we have

$$\lim_{m \to \infty} Cc(K_m \boxdot K_m) = \lim_{m \to \infty} \frac{2m^2 - 6m + 4}{4m^2 - 10m + 6} = \lim_{m \to \infty} \frac{2 - \frac{6}{m} + \frac{4}{m^2}}{4 - \frac{10}{m} + \frac{6}{m^2}} = \frac{1}{2}.$$

□

4 Conclusion

In this paper we were able to generate some useful formulas for the global clustering coefficient of the tensor product $K_m \otimes K_n$ and the cartesian product $K_m \boxdot K_n$. It was interesting to see that as m increases without bound, $Cc(K_m \otimes K_m) \to 1$ while $Cc(K_m \boxdot K_m) \to 1/2$. There might be other parameterized graphs resulting from other graph binary operations and even some unary operations that may exhibit similar interesting properties. Their determination can be a subject of a separate investigation.

Acknowledgement

The authors would like to acknowledge the valuable comments and inputs made by the anonymous referees. In addition, the first author acknowledges with gratitude the scholarship support granted to him by the Department of Science and Technology, Philippines.

Competing Interests

Authors have declared that no competing interests exist.

References

[1] Takahashi N. On clustering coefficients of graphs with fixed numbers of vertices and edges. In: 2009 European Conference on Circuit Theory and Design, 2009;814-817.

[2] Li Y, Shang Y, Yang Y. Clustering coefficients of large networks. Information Sciences. 2017;382:350-358.

[3] Watts DJ, Strogatz SH. Collective dynamics of small-world networks. Nature. 1998;393(6684):440-442.

[4] Damalerio RJM, Eballe RG. Clustering coefficient of the tensor product of graphs. Asian Journal of Mathematics. 2022;18(6):36-42.
Available: https://doi.org/10.9734/arjom/2022/v18i630382

[5] Damalerio RJM, Eballe RG. Triangular index of some graph products. Applied Mathematical Sciences. 2021;15(12):587-594.

[6] Suri S, Vassilvitskii S. Counting triangles and the curse of the last reducer. In: Proceedings of the 20th International Conference on World Wide Web; 2011. Available: https://doi:10.1145/1963405.1963491

[7] Acosta HR, Eballe RG, Cabahug IS Jr. Downhill domination in the tensor product of graphs. International Journal of Mathematical Analysis. 2019;13(12):555-564.

[8] Eballe R, Aldema R, Paluga EM, Rulete RF, Jamil FP. Global defensive alliances in the join, corona and composition of graphs. Ars Comb. 2012;(107):225-245.

[9] Eballe RG, Canoy SR Jr. The essential cutset number and connectivity of the join and composition of graphs. Utilitas Mathematica. 2011;84:257-263.

[10] Eballe RG, Llido EM, Nocete MT. Vertex independence in graphs under some binary operations. Matimyas Matematika. 2007;30(1):37-40.

[11] Chartrand G, Lesniak L, Zhang P. Graphs and digraphs (6th ed.), Chapman and Hall;2015. Available:https://doi.org/10.1201/b19731

[12] Harary F. Graph Theory. Addisson-Wesley Publishing Company, Inc. USA; 1969. Available:https://doi.org/10.21236/ad0705364

Peer-review history:
The peer review history for this paper can be accessed here (Please copy paste the total link in your browser address bar)
https://www.sdiarticle5.com/review-history/87167