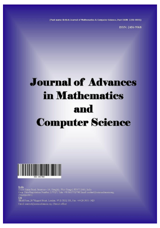

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Study on Two New Numbers and Polynomials Numbers and Polynomials Arising from the Fermionic p-adic Integral on $\mathbb{Z}_p$

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Abstract

p-adic analysis and their applications is used p-adic distributions, p-adic measure, p-adic integrals, p-adic L-function and other generalized functions. In addition, among the many ways to investigate and construct generating functions for special polynomials and numbers, one of the most important techniques is the p-adic Fermionic integral over $\mathbb{Z}_p$. In this paper, we introduce new numbers and polynomials arising from the Fermionic p-adic integral on $\mathbb{Z}_p$. First, we introduce new numbers and polynomials as one of generalizations of Changhee numbers and polynomials of order r ($r \in \mathbb{N}$), which are called the generalized Changhee numbers and polynomials. We explore some interesting identities and explicit formulas of these numbers and polynomials. Second, we define new numbers and polynomials as one of generalizations of Catalan numbers and polynomials of order r ($r \in \mathbb{N}$), which are called the generalized Catalan numbers and polynomials. We also study some combinatorial identities and explicit formulas of these numbers and polynomials.

Keywords: *p-adic Fermionic integral on $\mathbb{Z}_p$; the Catalan numbers and polynomials of order r; the Changhee numbers of the first kind of order r; Euler numbers and polynomials; the Apostrol Euler numbers.*

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1 Introduction

Initiated by Kurt Hensel (1861-1941) at the end of the 19th century, the p-adic numbers have recently been applied in physics, mathematics, and engineering in other parts of the natural sciences. In particular, the p-adic analysis and their applications utilize p-adic distributions and p-adic measure, p-adic integrals, p-adic L-function, and other generalized functions. Among these, the p-adic integral and its applications are very important in finding solutions to special (differential) equations, real problems in both physics and engineering ([1-20]). In addition, There are many methods and techniques for investigating and constructing generating functions for special polynomials and numbers ([1-3, 5, 11-13, 17, 21-30]). One of the most important techniques is the p-adic Fermionic integral on $\mathbb{Z}_p$. In [9], Kim constructed the p-adic q-Volkenborn integration. When $q = -1$, it is called the p-adic Fermionic integral on $\mathbb{Z}_p$ ([10]). In this paper, we introduce two new numbers and polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$. For $p \equiv 1 (\text{mod } 2)$, $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, $a \in \mathbb{Q}^+$, $b \in \mathbb{Q} - \{0\}$ with $(a,p) = (b,p) = 1$, we first introduce new numbers $A_n^{(r)}(a,b)$ and polynomials $A_n^{(r)}(a,b|x)$ of a generalization of Changhee numbers and polynomials of order r ($r \in \mathbb{N}$), respectively. We explore some interesting identities and explicit formulas of these numbers and polynomials. Second, we define new numbers $W_n^{(r)}(a,b)$ and polynomials $W_n^{(r)}(a,b|x)$, respectively, for one of generalizations of Catalan numbers and polynomials of order r ($r \in \mathbb{N}$). We also invesgete some interesting properties and explicit formulas of these numbers and polynomials.

Let p be a prime number with $p \equiv 1$ (mod 2). Throughout this paper, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let$|\cdot|_p$ be the p-adic norm with $|p|_p = \frac{1}{p}$.

For a $\mathbb{C}_p$-valued continuous function f on $\mathbb{Z}_p$, Kim [9, 10] introduced the p-adic fermionic integral on $\mathbb{Z}_p$ as follows:

$$\begin{aligned} I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)\mu_{-1}(x + p^{\mathbb{N}}\mathbb{Z}_p) \\ &= \lim_{\mathbb{N}\to\infty} \sum_{x=0}^{p^{\mathbb{N}}-1} f(x)(-1)^x, \quad \text{(see [4, 10, 11, 18])}. \end{aligned} \tag{1.1}$$

Let $f_n(x) = f(x+n)$ for $n \in \mathbb{N}$. From (1.5), we observe that

$$I_{-1}(f_n) + (-1)^{n-1} I_{-1}(f) = 2\sum_{l=0}^{n-1} (-1)^{n-1-l} f(l), \quad \text{(see [4, 10, 11, 18])}. \tag{1.2}$$

In (1.2), when $n = 1$, we have

$$I_{-1}(f_1) + I_{-1}(f) = 2f(0). \tag{1.3}$$

From (1.3), for $r \in \mathbb{N}$, Kim-Kim introduced the Changhee numbers $Ch_n^{(r)}$ and polynomials $Ch_n^{(r)}(x)$ of the first kind of order r, respectively, as follows:

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)_n d\mu_{-1}(x_1) \cdots d_{\mu-1}(x_r) = Ch_n^{(r)}, \quad \text{(see [7])}, \tag{1.4}$$

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1+t)^{x_1+\cdots+x_r+x}d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r) = \left(\frac{2}{2+t}\right)^r(1+t)^x = \sum_{n=0}^{\infty} Ch_n^{(r)}(x)\frac{t^n}{n!}, \qquad \text{(see [7])}. \tag{1.5}$$

When $x = 0$, $Ch_n^{(r)} = Ch_n(0)$, which are called the Changhee numbers of order r.

When $r = 1$, $Ch_n = Ch_n^{(1)}$ and $Ch_n(x) = Ch_n^{(1)}(x)$, which are called the Changhee numbers and Changhee polynomials, respectively.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, from (1.3), we have the Catalan numbers C_n given by the generating function

$$\int_{\mathbb{Z}_p}(1-4t)^{\frac{x}{2}}d\mu_{-1}(x) = \frac{2}{\sqrt{1-4t}+1} = \sum_{n=0}^{\infty} C_n t^n, \qquad \text{(see [11])}, \tag{1.6}$$

and the Catalan number $C_n^{(r)}$ of order r ($r \in \mathbb{N}$) given by the generating function

$$\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(1-4t)^{\frac{1}{2}(x_1+x_2+\cdots+x_r)}d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r) = \left(\frac{2}{\sqrt{1-4t}+1}\right)^r = \sum_{n=0}^{\infty} C_n^{(r)} t^n. \tag{1.7}$$

The p-adic logarithm and exponential function are given by the following infinite series:

$$\log(1+t) = -\sum_{n=1}^{\infty}\frac{(-t)^n}{n}, \qquad (s \in \mathbb{C}_p, \ \ |t|_p < 1),$$

and

$$e^t = \sum_{n=1}^{\infty}\frac{t^n}{n!}, \qquad (s \in \mathbb{C}_p, \ \ |t|_p < p^{\frac{p}{p-1}}).$$

From (1.3), the Euler polynomials are given by

$$\int_{\mathbb{Z}_p} e^{t(y+x)} \ d_{\mu-1}(y) = \frac{2}{e^t+1}e^{xt} = \sum_{n=0}^{\infty} E_n(x)\frac{t^n}{n!}, \qquad \text{(see [5, 8, 10])}. \tag{1.8}$$

When $x = 0$, $E_n = E_n(0)$, which are called the Euler numbers.

From (1.3), we get

$$\int_{\mathbb{Z}_p} x^n \ d_{\mu-1}(y) = E_n \ \text{ and } \int_{\mathbb{Z}_p}(y+x)^n \ d_{\mu-1}(y) = E_n(x), \qquad \text{(see [5, 8, 10])}. \tag{1.9}$$

Let T_p be the p-adic locally constant space defined by $T_p = \bigcup_{n\geq 1} = \lim_{n\to\infty} C_{p^n}$, $(n \in \mathbb{N})$, where $C_{p^n} = \{\mu \mid \mu^{p^n} = 1\}$. For $\mu \in T_p$ and $t \in \mathbb{C}_p$, the Apostol-Euler polynomials $\mathcal{E}_n(x;\mu)$ were introduced by

$$\frac{2e^{xt}}{\mu e^t+1} = \sum_{n=0}^{\infty}\mathcal{E}_n(x;\mu)\frac{t^n}{n!}, \qquad \text{(see [3, 15, 18])}, \tag{1.10}$$

when $x = 0$, $\mathcal{E}_n(\mu) = 2^n \mathcal{E}_n(\frac{1}{2};\mu)$, which are called the Apostrol-Euler numbers.

Obviously, when $\mu = 1$, $\mathcal{E}_n(x;1) = E_n(x)$.

The Euler polynomials of order r $(r \in \mathbb{N})$ are given by the generating function

$$\left(\frac{2}{e^t+1}\right)^r e^{xt} = \sum_{n=0}^{\infty} E_n^{(r)}(x)\frac{t^n}{n!}, \tag{1.11}$$

when $x = 0$, $E_n^{(r)} = E_n^{(r)}(0)$, which are called the Euler numbers of order r.

For $n \geq 0$, the Stirling numbers of second kind are defined by

$$(x)_n = \sum_{l=0}^{n} S_1(n,l)x^l, \quad \text{and} \quad \frac{1}{k!}(\log(1+t))^k = \sum_{n=k}^{\infty} S_1(n,k)\frac{t^n}{n!}, \quad \text{(see [1, 2])}. \tag{1.12}$$

and

$$x^n = \sum_{l=0}^{n} S_2(n,l)(x)_l, \quad \text{and} \quad \frac{1}{k!}(e^t-1)^k = \sum_{n=k}^{\infty} S_2(n,k)\frac{t^n}{n!}, \quad \text{(see [1, 2])}, \tag{1.13}$$

where $(x)_n = x(x-1)(x-2)\cdots(x-n+1)$ and $(x)_0 = 1$.

2 The Generalized Changhee Numbers and Polynomials Arising from the Fermionic p-adic Integral on $\mathbb{Z}_p$

In this section, we study new numbers of polynomials as one generalization of Changhee numbers and polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$, called the generalized Changhee numbers and polynomials. We derive many properties of them.

Throughout this paper, assume that $p \equiv 1 (\text{mod } 2)$, $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, $a \in \mathbb{Q}^+$, b$\in \mathbb{Q} - \{0\}$ with $(a,p) = 1 = (b,p)$ and $(b,t) = 1$, where (m,n) is the greatest common divisor of m and n.

Let $f(x) = a + bt$. From (1.3), we observe that

$$\int_{\mathbb{Z}_p} (a+bt)^x d\mu_{-1}(x) = \frac{2}{(a+1)+bt} = \sum_{n=0}^{\infty} A_n(a,b)t^n. \tag{2.1}$$

In particular, when $a = 1,\ b = 1$, the generating function of Changhee numbers of the first kind are given by

$$\int_{\mathbb{Z}_p} (1+t)^x d\mu_{-1}(x) = \frac{2}{2+t} \quad \text{and} \quad n!A_n(1,1) = Ch_n. \tag{2.2}$$

When $a = 1,\ b = -1$, we get

$$\int_{\mathbb{Z}_p} (1-t)^x d\mu_{-1}(x) = \frac{2}{2-t} \quad \text{and} \quad n!A_n(1,-1) = (-1)^n Ch_n. \tag{2.3}$$

Theorem 1. For $a \in \mathbb{Q}^+$, $b \in \mathbb{Q} - \{0\}$ with $(a, p) = 1 = (b, p)$ and $(b, t) = 1$, we have

$$A_n(a, b) = \frac{b^n}{n!a^n} \int_{\mathbb{Z}_p} (x)_n a^x \; d\mu_{-1}(x) \quad \text{and,}$$

and

$$\int_{\mathbb{Z}_p} (x)_n a^x \; d\mu_{-1}(x) = \frac{2(-1)^n}{n!a^n(a+1)^{n+1}}.$$

Proof. From (1.3), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (a+bt)^x d\mu_{-1}(x) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{x}{n} a^{x-n} b^n \; d\mu_{-1}(x) t^n \\ &= \sum_{n=0}^{\infty} \frac{1}{n!a^n} \int_{\mathbb{Z}_p} (x)_n a^x b^n \; d\mu_{-1}(x) t^n.\end{aligned} \tag{2.4}$$

On the other hand, we get

$$\frac{2}{(a+1)+bt} = \frac{2}{(a+1)(1+\frac{b}{a+1}t)} = \frac{2}{a+1} \sum_{n=0}^{\infty} \left(-\frac{b}{a+1}\right)^n t^n. \tag{2.5}$$

By comparing the coefficients of (2.4) and (2.5), we get the desired result.

□

Remark. By (1.1), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (-1)^x x^k d\mu_{-1}(x) &= \lim_{N\to\infty} \int_{\bigcup_{x=0}^{p^N-1}(x+p^N\mathbb{Z}_p)} (-1)^x x^k \; d\mu_{-1}(x) \\ &= \lim_{N\to\infty} \sum_{x=0}^{p^N-1} (-1)^x x^k \mu_{-1}(x+p^N\mathbb{Z}_p) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} x^k = 0.\end{aligned} \tag{2.6}$$

When $a = -1$, combining (1.1) with (2.6), we have

$$\begin{aligned}\int_{\mathbb{Z}_p} (-1+bt)^x d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} e^{x\log(1-bt)} (-1)^x \; d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} (-1)^x x^l \; d\mu_{-1}(x) \frac{1}{l!} (\log(1-bt))^l \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{(-1)^n b^n}{n!} S_1(n, l) \int_{\mathbb{Z}_p} (-1)^x x^k \; d\mu_{-1}(x) t^n = 0,\end{aligned}$$

Theorem 2. For $a = 1$, $b \in \mathbb{Q} - \{0\}$ with $(b, p) = 1$ and $(b, t) = 1$, we have

$$A_n(1, b) = \frac{b^n}{n!} \sum_{l=0}^{n} S_1(n, l) E_l,$$

where E_n are the Euler numbers.

Proof. From (1.12) and (2.1), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty} A_n(1,b)t^n &= \int_{\mathbb{Z}_p} (1+bt)^x d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} x^l \frac{1}{l!}(\log(1+bt))^l d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} x^l \sum_{n=l}^{\infty} S_1(n,l)\frac{b^n}{n!}t^l d\mu_{-1}(x) \\ &= \sum_{n=0}^{\infty}\sum_{l=0}^{n} \frac{S_1(n,l)b^n}{n!}\int_{\mathbb{Z}_p} x^l d\mu_{-1}(x)t^n.\end{aligned} \tag{2.7}$$

By comparing the coefficients of both sides of (2.7), we get the desired result. □

Theorem 3. For $b \in \mathbb{Q}^+$ with $(b,p)=1$ and $(b,t)=1$, we have,

$$\sum_{m=0}^{n} m! A_m(b,b) S_2(n,m) = \mathcal{E}_n(b),$$

where $\mathcal{E}_n(b)$ are the Apostrol-Euler numbers.

Proof. Let

$$\sum_{n=0}^{\infty} A_n(b,b)t^n = \int_{\mathbb{Z}_p} (b+bt)^x d\mu_{-1}(x). \tag{2.8}$$

Replacing t by e^t-1 in (2.8), from (1.3) and (1.10), the left-hand side of (2.8) is

$$\int_{\mathbb{Z}_p} (b+b(e^t-1))^x d\mu_{-1}(x) = \int_{\mathbb{Z}_p} (be^t)^x d\mu_{-1}(x) = \frac{2}{be^t+1} = \sum_{n=0}^{\infty} \mathcal{E}_n(b)\frac{t^n}{n!}. \tag{2.9}$$

By (1.13), the right-hand side of (2.8) is

$$\begin{aligned}\sum_{m=0}^{\infty} A_m(b,b)(e^t-1)^m &= \sum_{m=0}^{\infty} m! A_m(b,b)\frac{(e^t-1)^m}{m!} \\ &= \sum_{m=0}^{\infty} m! A_m(b,b)\sum_{n=m}^{\infty} S_2(n,m)\frac{t^n}{n!} \\ &= \sum_{n=0}^{\infty}\sum_{m=0}^{n} m! A_m(b,b) S_2(n,m)\frac{t^n}{n!}.\end{aligned} \tag{2.10}$$

By comparing the coefficients of (2.9) and (2.10), we get the desired identity. □

For $a \in \mathbb{Q}^+$, $b \in \mathbb{Q}-\{0\}$ with $(a,p)=1=(b,p)$ and $(b,t)=1$, we consider the generating function of $A_n(a,b|x)$ which are derived from the Fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} (a+bt)^{y+x}\ d\mu_{-1}(y) = \sum_{n=0}^{\infty} A_n(a,b|x)t^n. \tag{2.11}$$

When $x = 0$, $A_n(a, b) = A_n(a, b|0)$. From (1.3), we have

$$\sum_{n=0}^{\infty} A_n(a, b|x)t^n = \frac{2}{(a+1)+bt}(a+bt)^x. \tag{2.12}$$

We note that $n!A_n(1, 1|x) = Ch_n(x)$ and $A_n(1, -1|x) = \frac{(-1)^n}{n!}Ch_n(x)$.

Theorem 4. For $a \in \mathbb{Q}^+$, $b \in \mathbb{Q} - \{0\}$ with $(a, p) = 1 = (b, p)$ and $(b, t) = 1$, we have

$$A_n(a, b|x) = \frac{b^n}{n!a^n} a^x \int_{\mathbb{Z}_p} (y + x)_n a^y \; d\mu_{-1}(y).$$

In addition, we have

$$\int_{\mathbb{Z}_p} (y + x)_n a^y \; d\mu_{-1}(y) = \sum_{m=0}^{n} n!(-1)^{n-m} A_m(a, 1).$$

Proof. We observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (a + bt)^{y+x} \; d\mu_{-1}(y) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{y+x}{n} a^{y+x-n} b^n \; d\mu_{-1}(y)t^n \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (y + x)_n a^y \frac{a^x b^n}{n!a^n} \; d\mu_{-1}(y)t^n.\end{aligned} \tag{2.13}$$

By comparing the coefficients of (2.11) and (2.13), we have the first identity.

In particular, when $b = 1$, we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p} (1 + t)^{y+x} a^y \; d\mu_{-1}(y) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{y+x}{n} a^y t^n \; d\mu_{-1}(y) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} (y + x)_n a^y \; d\mu_{-1}(y) \frac{t^n}{n!}.\end{aligned} \tag{2.14}$$

On the other hand, from (1.3), we get

$$\begin{aligned}\int_{\mathbb{Z}_p} (1 + t)^{y+x} a^y \; d\mu_{-1}(y) &= \frac{2}{(a+1)+t}(1 + t)^x = \sum_{m=0}^{\infty} A_m(a, 1)t^m \sum_{l=0}^{\infty} (-1)^l t^l \\ &= \sum_{n=0}^{\infty} \sum_{m=0}^{n} n!(-1)^{n-m} A_m(a, 1) \frac{t^n}{n!}.\end{aligned} \tag{2.15}$$

By comparing the coefficients of (2.14) and (2.15), we have the second identity. □

In the same way as Theorem 2 and 3, we have the following theorem.

Theorem 5. For $b \in \mathbb{Q} - \{0\}$ with $(b, p) = 1$ and $(b, t) = 1$, we have

$$A_n(1, b|x) = \frac{b^n}{n!} \sum_{l=0}^{n} S_1(n, l)E_l(x)$$

and

$$\sum_{m=0}^{n} m!A_m(b,b|x)S_2(n,m) = \mathcal{E}_n(b|x),$$

where $E_n(x)$ and $\mathcal{E}_n(b|x)$ are the Euler polynomials and the Apostrol-Euler polynomials.

Theorem 6. For $b \in \mathbb{Q} - \{0\}$, we have

$$A_n(1,b|x) = \sum_{l=0}^{n}\sum_{j=0}^{l} \frac{(-1)^{n-l}b^n}{l!2^{n-l}}x^j.$$

Proof. From (1.12), we observe that

$$\begin{aligned}(1+bt)^x &= \sum_{j=0}^{\infty} x^j \frac{1}{j!}(\log(1+bt))^j \\ &= \sum_{j=0}^{\infty} x^j \sum_{l=j}^{\infty} S_1(l,j)\frac{b^l t^l}{l!} = \sum_{l=0}^{\infty}\Big(\sum_{j=0}^{l} S_1(l,j)\frac{b^l}{l!}x^j\Big)t^l.\end{aligned} \tag{2.16}$$

On the other hand, we have

$$\begin{aligned}\frac{2}{2+bt}(1+bt)^x &= \sum_{i=0}^{\infty}\Big(-\frac{b}{2}\Big)^i t^i \sum_{l=0}^{\infty}\Big(\sum_{j=0}^{l}\frac{b^l}{l!}x^j\Big)t^l \\ &= \sum_{n=0}^{\infty}\Big(\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{(-1)^{n-l}b^n}{l!2^{n-l}}x^j\Big)t^n.\end{aligned} \tag{2.17}$$

By comparing the coefficients of (2.16) and (2.17), we get the desired result. □

For $r \in \mathbb{N}$, $a \in \mathbb{Q}^+$, and $b \in \mathbb{Q}-\{0\}$ with $(a,p) = 1 = (b,p)$ and $(b,t) = 1$, we consider the generating functions of $A_n^{(r)}(a,b)$ and $A_n^{(r)}(a,b|x)$ of order r, which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$, respectively as follows:

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{x_1+x_2+\cdots+x_r}\ d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r) \\ &\qquad = \Big(\frac{2}{(a+1)+bt}\Big)^r = \sum_{n=0}^{\infty}A_n^{(r)}(a,b)t^n,\end{aligned} \tag{2.18}$$

and

$$\begin{aligned}&\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{x_1+x_2+\cdots+x_r+x}\ d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r) \\ &\qquad = \Big(\frac{2}{(a+1)+bt}\Big)^r(a+bt)^x = \sum_{n=0}^{\infty}A_n^{(r)}(a,b|x)t^n.\end{aligned} \tag{2.19}$$

It easy to see that $n!A_n^{(r)}(1,1) = Ch_n^{(r)}$ and $n!A_n^{(r)}(1,1|x) = Ch_n^{(r)}(x)$.

Theorem 7. For $a \in \mathbb{Q}^+$ and $b \in \mathbb{Q} - \{0\}$ with $(a,p) = (b,p) = 1$ and $(b,t) = 1$, we have

$$A_n^{(r)}(a,b|x) = \left(\frac{b}{a}\right)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n a^{x_1+\cdots+x_r+x} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

In particular, when $x = 0$, we have

$$A_n^{(r)}(a,b) = \left(\frac{b}{a}\right)^n \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r)_n a^{x_1+\cdots+x_r} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r).$$

Proof. We observe that

$$\begin{aligned} &\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (a+bt)^{x_1+\cdots+x_r+x} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) \\ &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \binom{x_1 + \cdots + x_r + x}{n} a^{x_1+\cdots+x_r-n} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) b^n t^n \\ &= \frac{b^n}{a^n} \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} (x_1 + \cdots + x_r + x)_n a^{x_1+\cdots+x_r} \, d\mu_{-1}(x_1) \cdots d\mu_{-1}(x_r) t^n. \end{aligned} \tag{2.20}$$

Combining (2.19) with (2.20), we get the desired result. □

Theorem 8. For $r \in \mathbb{N}$, $a \in \mathbb{Q}^+$, and $b \in \mathbb{Q} - \{0\}$ with $(a,p) = 1 = (b,p)$ and $(b,t) = 1$, we have

$$A_n^{(r)}(a,b) = \sum_{j_1+j_2+\cdots+j_r=n} \binom{n}{j_1 j_2 \cdots j_r} A_{j_1}(a,b) A_{j_2}(a,b) \cdots A_{j_r}(a,b).$$

Proof. We observe that

$$\left(\frac{2}{(a+1)+bt}\right)^r = \sum_{n=0}^{\infty} \left(\sum_{j_1+j_2+\cdots+j_r=n} \binom{n}{j_1 j_2 \cdots j_r} A_{j_1}(a,b) A_{j_2}(a,b) \cdots A_{j_r}(a,b) \right) \frac{t^n}{n!}. \tag{2.21}$$

From (2.21), we ge the desired identity. □

Theorem 9. For $r \in \mathbb{N}$, $b \in \mathbb{Q} - \{0\}$ with $(b,p) = 1$ and $(b,t) = 1$, we have

$$A_n^{(r)}(1,b|x) = \sum_{l=0}^{n} \sum_{j=0}^{l} \frac{b^l}{l!} S_1(l,j) A_{n-l}^{(r)}(1,b) x^j.$$

In addition, when $x = 0$, we have

$$A_n^{(r)}(1,b) = \sum_{l=0}^{n} \frac{b^l}{l!} A_{n-l}^{(r)}(1,b) x^j.$$

Proof. From (1.12) and (2.19), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} A_n^{(r)}(1,b|x) t^n &= \left(\frac{2}{2+bt}\right)^r (1+bt)^x = \sum_{m=0}^{\infty} A_m^{(r)}(1,b) t^m \sum_{l=0}^{\infty} \left(\sum_{j=0}^{l} S_1(l,j) \frac{b^l}{l!} x^j \right) t^l \\ &= \sum_{n=0}^{\infty} \left(\sum_{l=0}^{n} \sum_{j=0}^{l} \frac{b^l}{l!} S_1(l,j) A_{n-l}^{(r)}(1,b) x^j \right) t^n. \end{aligned} \tag{2.22}$$

By comparing the coefficients of both sides of (2.22), we get the desired result.

□

3 The Generalized Catalan Numbers and Polynomials Arising from the Fermionic p-adic Integral on $\mathbb{Z}_p$

In this section, we study new numbers of polynomials as one generalization of Catalan numbers and polynomials which derived from the Fermionic p-adic integral on $\mathbb{Z}_p$, called the generalized Catalan numbers and polynomials. We also explore interesting properties.

For $t \in \mathbb{C}_p$ with $|t|_p < p^{-\frac{1}{p-1}}$, $a \in \mathbb{Q}^+$, and $b \in \mathbb{Q} - \{0\}$ with $(a, p) = 1 = (b, p)$ and $(b, t) = 1$, let $f(x) = a + bt$.

From (1.3), we observe that

$$\int_{\mathbb{Z}_p} (a+bt)^{\frac{x}{2}} d\mu_{-1}(x) = \frac{2}{\sqrt{a+bt}+1} = \sum_{n=0}^{\infty} W_n(a,b)t^n. \tag{3.1}$$

In particular, when $a = 1,\ b = -4$, we get the generating function of Catalan numbers as follows:

$$\int_{\mathbb{Z}_p} (1-4t)^{\frac{x}{2}} d\mu_{-1}(x) = \frac{2}{\sqrt{1-4t}+1} \quad \text{and} \quad W_n(1,-4) = C_n. \tag{3.2}$$

When $a = 1,\ b = 4$, we get

$$\int_{\mathbb{Z}_p} (1+4t)^{\frac{x}{2}} d\mu_{-1}(x) = \frac{2}{\sqrt{1+4t}+1} \quad \text{and} \quad W_n(1,4) = (-1)^n C_n. \tag{3.3}$$

To proof of next theorem, we observe that

$$\begin{aligned}
\sqrt{1+bt} &= \sum_{n=0}^{\infty} \binom{\frac{1}{2}}{n} t^n = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)_n \frac{1}{n!} b^n t^n \\
&= \sum_{n=0}^{\infty} \frac{\frac{1}{2}(\frac{1}{2}-1)(\frac{1}{2}-2)\cdots(\frac{1}{2}-n+1)}{n!} b^n t^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 1\cdot 3\cdot 5\cdots(2n-3)}{n!2^n} b^n t^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1} 1\cdot 2\cdot 3\cdot 4\cdots(2n-2)(2n-3)(2n-1)(2n)}{n!2^n 2\cdot 4\cdot 6\cdots(2n-2)(2n-1)(2n)} b^n t^n \\
&= \sum_{n=0}^{\infty} \frac{(-1)^{n-1}(2n)!}{n!4^n(2n-1)(n!)} t^n = \sum_{n=0}^{\infty} \frac{(-1)^{n-1}}{4^n(2n-1)} \binom{2n}{n} b^n t^n.
\end{aligned} \tag{3.4}$$

Theorem 10. For $b \in \mathbb{Q} - \{0\}$ with $(b, p) = 1$ and $(b, t) = 1$, we have

$$n! W_n(1,b) = b^n \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x),$$

and

$$\int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x) = \frac{2(-1)^{n+1}}{4^{n+1}(2n+1)} \binom{2(n+1)}{n+1}.$$

Proof. First, we observe that

$$\begin{aligned} \int_{\mathbb{Z}_p} (1+bt)^{\frac{x}{2}} d\mu_{-1}(x) &= \sum_{n=0}^{\infty} \int_{\mathbb{Z}_p} \binom{\frac{x}{2}}{n} b^n d\mu_{-1}(x) t^n \\ &= \sum_{n=0}^{\infty} \frac{b^n}{n!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)_n d\mu_{-1}(x) t^n. \end{aligned} \tag{3.5}$$

Combining (3.1) and (3.5), we get the first identity.

From (3.4), we get

$$\begin{aligned} \frac{2}{1+\sqrt{1+bt}} &= \frac{2(1-\sqrt{1+bt})}{bt} = \frac{2}{bt} \sum_{n=1}^{\infty} \frac{b^n(-1)^n}{4^n(2n-1)} \binom{2n}{n} t^n \\ &= 2 \sum_{n=0}^{\infty} \frac{b^n(-1)^{n+1}}{4^{n+1}(2n+1)} \binom{2(n+1)}{n+1} t^n. \end{aligned} \tag{3.6}$$

By comparing the coefficients of (3.5) and (3.6), we get the second identity.

□

Theorem 11. For $b \in \mathbb{Q} - \{0\}$ with $(b,p) = 1$ and $(b,t) = 1$, we have

$$W_n(1,b) = \sum_{l=0}^{n} \frac{b^n}{n! 2^l} S_1(n,l) E_l,$$

where E_n are the Euler numbers.

Proof. From (1.9) and (1.12), we observe that

$$\begin{aligned} \sum_{n=0}^{\infty} W_n(1,b) t^n &= \int_{\mathbb{Z}_p} (1+bt)^{\frac{x}{2}} \; d\mu_{-1}(x) \\ &= \int_{\mathbb{Z}_p} e^{\frac{x}{2}\log(1+bt)} \; d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)^l \frac{1}{l!} (\log(1+bt))^l \; d\mu_{-1}(x) \\ &= \sum_{l=0}^{\infty} \sum_{n=l}^{\infty} S_1(n,l) \frac{b^n}{n!} \int_{\mathbb{Z}_p} \left(\frac{x}{2}\right)^l \; d\mu_{-1}(x) t^n \\ &= \sum_{n=0}^{\infty} \sum_{l=0}^{n} \frac{b^n}{n! 2^l} S_1(n,l) E_l t^n. \end{aligned} \tag{3.7}$$

By comparing the coefficients of both sides of (3.7), we get the desired result.

□

The next theorem is the inverse formula of Theorem 11.

Theorem 12. For $b \in \mathbb{Q} - \{0\}$ with $(b, p) = 1$ and $(b, t) = 1$, we have

$$\sum_{m=0}^{n} \frac{m!2^n}{b^m} S_2(n, m) W_m(1, b) = E_n,$$

where E_n are the ordinary Euler numbers.

Proof. Let

$$\frac{2}{\sqrt{1+bt}+1} = \sum_{n=0}^{\infty} W_n(1, b) t^n. \tag{3.8}$$

By replacing t by $\frac{1}{b}(e^{2t} - 1)$ in (3.8), by (1.9), the left-hand side of (3.8) is

$$\frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}. \tag{3.9}$$

On the other hand, from (1.13), the right-hand side of (3.8) is

$$\begin{aligned}\sum_{m=0}^{\infty} W_m(1, b)\left(\frac{1}{b}(e^{2t} - 1)\right)^m &= \sum_{m=0}^{\infty} \frac{m!}{b^m} W_m(1, b) \sum_{n=m}^{\infty} S_2(n, m) \frac{2^n t^n}{n!} \\ &= \sum_{n=0}^{\infty} \left(\sum_{m=0}^{n} \frac{m!2^n}{b^m} S_2(n, m) W_m(1, b) \right) \frac{t^n}{n!}.\end{aligned} \tag{3.10}$$

By comparing the coefficients of (3.9) and (3.10), we have the desired result. □

For $a \in \mathbb{Q}^+$ and $b \in \mathbb{Q} = \{0\}$, we consider the generating function of $W_n(a, b|x)$ which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$ as follows:

$$\int_{\mathbb{Z}_p} (a + bt)^{\frac{y+x}{2}} \, d\mu_{-1}(y) = \sum_{n=0}^{\infty} W_n(a, b|x) t^n. \tag{3.11}$$

When $x = 0$, $W_n(a, b) = W_n(a, b|0)$. From (1.3), we have

$$\sum_{n=0}^{\infty} W_n(a, b|x) t^n = \frac{2}{\sqrt{a+bt}+1} (a + bt)^{\frac{x}{2}}. \tag{3.12}$$

We note that $W_n(1, -4|x) = C_n(x)$ and $W_n(1, 4|x) = (-1)^n C_n(x)$.

Theorem 13. For $a \in \mathbb{Q}^+$, and $b \in \mathbb{Q} - \{0\}$ with $(a, p) = 1 = (b, p)$ and $(b, t) = 1$,

$$W_n(a, b|x) = \frac{b^n}{n!a^n} a^{\frac{x}{2}} \int_{\mathbb{Z}_p} \left(\frac{y+x}{2}\right)_n a^{\frac{y}{2}} \, d\mu_{-1}(y),$$

and

$$W_n(a, b|x) = \sum_{k=0}^{n} \frac{b^n}{n!a^n 2^k} a^{\frac{x}{2}} S_1(n, k) \mathcal{E}_k(x; a^{\frac{1}{2}}),$$

where $\mathcal{E}_k(x; \mu)$ are the Apostrol-Euler polynomials.

Proof. From (1.10) and (1.12), we observe that

$$\begin{aligned}\int_{\mathbb{Z}_p}(a+bt)^{\frac{y+x}{2}}\,d\mu_{-1}(y) &= \sum_{n=0}^{\infty}\int_{\mathbb{Z}_p}\binom{\frac{y+x}{2}}{n}a^{\frac{y+x}{2}-n}b^n t^n\,d\mu_{-1}(y)\\ &= \sum_{n=0}^{\infty}\frac{b^n}{n!a^n}a^{\frac{x}{2}}\int_{\mathbb{Z}_p}\left(\frac{y+x}{2}\right)_n a^{\frac{y}{2}}\,d\mu_{-1}(y)t^n\\ &= \sum_{n=0}^{\infty}\frac{b^n}{n!a^n}a^{\frac{x}{2}}\sum_{k=0}^{n}S_1(n,k)\frac{1}{2^k}\int_{\mathbb{Z}_p}(y+x)^k a^{\frac{y}{2}}\,d\mu_{-1}(y)t^n\\ &= \sum_{n=0}^{\infty}\sum_{k=0}^{n}\frac{b^n}{n!a^n2^k}a^{\frac{x}{2}}S_1(n,k)\mathcal{E}_k(x;a^{\frac{1}{2}})t^n.\end{aligned} \tag{3.13}$$

Combining (3.11) with (3.13), we attain the desired result. □

For $r\in\mathbb{N}$, we consider the generating functions of $W(a,b)$ and $W(a,b|x)$ of order r, which are derived from the multivariate Fermionic p-adic integral on $\mathbb{Z}_p$, respectively as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{x_1+x_2+\cdots+x_r}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ =\left(\frac{2}{\sqrt{a+bt}+1}\right)^r=\sum_{n=0}^{\infty}W_n^{(r)}(a,b)t^n,\end{aligned} \tag{3.14}$$

and

$$\begin{aligned}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}(a+bt)^{\frac{x_1+x_2+\cdots+x_r+x}{2}}\,d\mu_{-1}(x_1)d\mu_{-1}(x_2)\cdots d\mu_{-1}(x_r)\\ =\left(\frac{2}{\sqrt{a+bt}+1}\right)^r(a+bt)^{\frac{x}{2}}=\sum_{n=0}^{\infty}W_n^{(r)}(a,b|x)t^n.\end{aligned} \tag{3.15}$$

From (1.7), we note that $W_n^{(r)}(1,-4)=C_n^{(r)}$ and $W_n^{(r)}(1,-4)=C_n^{(r)}(x)$.

The following theorem can be obtained in the same way as in Theorem 7.

Theorem 14. For $a\in\mathbb{Q}^+$ and $b\in\mathbb{Q}-\{0\}$ with $(a,p)=(b,p)=1$ and $(b,t)=1$, we have

$$W_n^{(r)}(a,b|x)=\left(\frac{b}{a}\right)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{x_1+\cdots+x_r+x}{2}\right)_n a^{\frac{x_1+\cdots+x_r+x}{2}}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).$$

In particular, when $x=0$, we have

$$W_n^{(r)}(a,b)=\left(\frac{b}{a}\right)^n\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}\left(\frac{x_1+\cdots+x_r+x}{2}\right)_n a^{\frac{x_1+\cdots+x_r}{2}}\,d\mu_{-1}(x_1)\cdots d\mu_{-1}(x_r).$$

Theorem 15. For $r\in\mathbb{N}$, $a\in\mathbb{Q}^+$, and $b\in\mathbb{Q}-\{0\}$ with $(a,p)=1=(b,p)$ and $(b,t)=1$, we have

$$W_n^{(r)}(a,b)=\sum_{j_1+j_2+\cdots+j_r=n}\binom{n}{j_1j_2\cdots j_r}W_{j_1}(a,b)W_{j_2}(a,b)\cdots W_{j_r}(a,b).$$

Proof. We observe that

$$\left(\frac{2}{\sqrt{a+bt}+1}\right)^r = \sum_{n=0}^{\infty}\left(\sum_{j_1+j_2+\cdots+j_r=n}\binom{n}{j_1 j_2\cdots j_r}W_{j_1}(a,b)W_{j_2}(a,b)\cdots W_{j_r}(a,b)\right)t^n. \quad (3.16)$$

Combining (3.14) with (3.16), we ge the desired identity.

□

Theorem 16. For $r \in \mathbb{N}$, $b \in \mathbb{Q}-\{0\}$ with $(b,p)=1$ and $(b,t)=1$, we have

$$W_n^{(r)}(1,b|x) = \sum_{l=0}^{n}\sum_{j=0}^{l}\frac{(-1)^{n-j}b^{n-j+l}}{l!2^{2n-j}}C_{n-j}S_1(l,j)x^j,$$

where C_n are the Catalan numbers.

Proof. From (1.12), we observe that

$$\begin{aligned}(1+bt)^{\frac{x}{2}} &= \sum_{j=0}^{\infty}\left(\frac{x}{2}\right)^j\frac{1}{j!}(\log(1+bt))^j \\ &= \sum_{j=0}^{\infty}\left(\frac{x}{2}\right)^j\sum_{l=j}^{\infty}S_1(l,j)\frac{b^l t^l}{l!} = \sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}S_1(l,j)\frac{b^l}{2^j l!}x^j\right)t^l.\end{aligned} \quad (3.17)$$

By (1.6) and (3.17), we have

$$\begin{aligned}\left(\frac{2}{1+\sqrt{1+bt}}\right)^r(1+bt)^{\frac{x}{2}} &= \left(\frac{2}{1+\sqrt{1-4(-\frac{b}{4}t)}}\right)^r(1+bt)^{\frac{x}{2}} \\ &= \sum_{m=0}^{\infty}C_m\left(-\frac{b}{4}\right)^m t^m\sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}S_1(l,j)\frac{b^l}{2^j l!}x^j\right)t^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{(-1)^{n-j}b^{n-j+l}}{l!2^{2n-j}}C_{n-j}S_1(l,j)x^j\right)t^n.\end{aligned} \quad (3.18)$$

By comparing the coefficients of (3.15) and (3.18), we get the desired result.

□

Theorem 17. For $r \in \mathbb{N}$, $b \in \mathbb{Q}-\{0\}$ with $(b,p)=1$ and $(b,t)=1$, we have

$$W_n^{(r)}(1,b|x) = \sum_{l=0}^{n}\sum_{j=0}^{l}\frac{b^l}{l!2^j}S_1(l,j)W_{n-l}^{(r)}(1,b)x^j.$$

Proof. From (1.12) and (3.15), we observe that

$$\begin{aligned}\sum_{n=0}^{\infty}W_n^{(r)}(1,b|x)t^n &= \sum_{m=0}^{\infty}W_m^{(r)}(1,b)t^m\sum_{l=0}^{\infty}\left(\sum_{j=0}^{l}S_1(l,j)\frac{b^l}{2^j l!}x^j\right)t^l \\ &= \sum_{n=0}^{\infty}\left(\sum_{l=0}^{n}\sum_{j=0}^{l}\frac{b^l}{l!2^j}S_1(l,j)W_{n-l}^{(r)}(1,b)x^j\right)t^n.\end{aligned} \quad (3.19)$$

By comparing the coefficients of both sides of (3.19), we get the desired result.

□

4 Conclusion

In this paper, we introduced two new numbers and polynomials derived from the (multivariate) Fermionic p-adic integral on $\mathbb{Z}_p$. One is the generalized Changhee numbers and polynomials $A_n^{(r)}(a,b|x)$ of order r ($r \in \mathbb{N}$) and the other is the generalized Catalan numbers and polynomials $W_n^{(r)}(a,b|x)$ of order r ($r \in \mathbb{N}$). In particular, we found that we could not generalize to two new numbers and polynomials derived from the Fermionic p-adic integral on $\mathbb{Z}_p$ (Section 2: Remark) when $a \in \mathbb{Q}^-$ (Section 2: Remark). From our definitions, we observed that $n!A_n^{(r)}(1,1|x) = Ch_n^{(r)}(x)$ and $W_n^{(r)}(1,-4|x) = C_n^{(r)}(x)$, where $Ch_n^{(r)}(x)$ and $C_n^{(r)}$ are the Changhee polynomials of order r and the Catalan polynomials of order r, respectively. In Section 2, we obtained relations of between the generalized Changhee polynomials (numbers) of order r and the Euler polynomials (numbers) of order r in Theorem 2 and 5. In particular, the Apostrol-Euler polynomials was expressed by the finite some of the Stirling numbers of the second kind and $A_n(b,b|x)$ in Theorem 5. In Section 3, we showed relations of between the generalized Catalan numbers and the Euler numbers in Theorem 11 and 12. In Theorem 13, the generalized Catalan polynomials was expressed by the finite sum of the Stirling numbers of the first kind and the Apostrol-Euler polynomials. In addition, we obtained various different explicit formulas. As is well known, the catalan numbers have many combinatorial applications. As a follow-up to this paper, some symmetric identities for these new numbers and polynomials are an example of good applications of these new numbers. We expect that there will be many applications by appropriately adjusting the variables a, b of these generalized new numbers. As a result, for future projects, we would like to conduct research into some potential applications of the numbers and polynomials derived in this paper.

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Ethics Approval and Consent to Participate

The author declare that there is no ethical problem in the production of this paper.

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Competing Interests

Author has declared that no competing interests exist.

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