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Generalized Fibonacci Numbers with Indices in Arithmetic Progression and Sum of Their Squares: The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this paper, closed forms of the sum formulas $\sum_{k=0}^{n} x^k W_{mk+j}^2$ for generalized Fibonacci numbers are presented. As special cases, we give sum formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas.

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1 Introduction

The Fibonacci numbers, Lucas numbers and their generalizations have many interesting properties and applications to almost every field such as architecture, nature, art, physics and engineering. The sequence of Fibonacci numbers $\{F_n\}_{n\geq 0}$ is defined by

$$F_n = F_{n-1} + F_{n-2}, \quad n \geq 2, \qquad F_0 = 0, \; F_1 = 1,$$

and the sequence of Lucas numbers $\{L_n\}_{n\geq 0}$ is defined by

$$L_n = L_{n-1} + L_{n-2}, \quad n \geq 2, \qquad L_0 = 2, \; L_1 = 1.$$

The generalization of Fibonacci sequence leads to several nice and interesting sequences. The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0, W_1; r, s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [1]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \qquad W_0 = a, W_1 = b, \quad n \geq 2 \tag{1.1}$$

where W_0, W_1 are arbitrary complex (or real) numbers and r, s are real numbers, see also Horadam [2,3,4] and Soykan [5].

For some specific values of a, b, r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r, s and initial values.

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS[6]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\dfrac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\dfrac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1, 2, 3, ...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

Now we define two special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \qquad G_0 = 0, G_1 = 1, \tag{1.2}$$

$$H_{n+2} = rH_{n+1} + sH_n, \qquad H_0 = 2, H_1 = r, \tag{1.3}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)},$$

$$H_{-n} = -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)},$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r, s) sequence $\{G_n(0, 1; r, s)\}_{n\geq 0}$ and Lucas (r, s) sequence $\{H_n(2, r; r, s)\}_{n\geq 0}$ are as follows:

1. $G_n(0, 1; 1, 1) = F_n$, Fibonacci sequence,
2. $H_n(2, 1; 1, 1) = L_n$, Lucas sequence,
3. $G_n(0, 1; 2, 1) = P_n$, Pell sequence,
4. $H_n(2, 2; 2, 1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0, 1; 1, 2) = J_n$, Jacobsthal sequence,
6. $H_n(2, 1; 1, 2) = j_n$, Jacobsthal-Lucas sequence.

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [7,8,9,10,11,12,13,14,15,16,17,18,19,20,21].

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [22,23,24,25,26,27,28,29]. For higher order Pell sequences, see [30,31,32,33,34,35].

We give the ordinary generating function $\sum_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.1. Suppose that $f_{W_n}(x) = \sum_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum_{n=0}^{\infty} W_n x^n$ is given by

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.4}$$

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{1.5}$$

The roots of characteristic equation are

$$\alpha = \frac{r + \sqrt{\Delta}}{2}, \quad \beta = \frac{r - \sqrt{\Delta}}{2}. \tag{1.6}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{aligned} \alpha + \beta &= r, \\ \alpha\beta &= -s, \\ (\alpha - \beta)^2 &= (\alpha + \beta)^2 - 4\alpha\beta = r^2 + 4s. \end{aligned}$$

1.1 Binet's Formula for the Distinct Roots Case

In this subsection, we assume that the roots α and β of characteristic equation (1.5) are distinct. Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.2 (Distinct Roots Case)**.** Binet's formula of generalized Fibonacci numbers is

$$W_n = \frac{b_1\alpha^n}{\alpha-\beta} + \frac{b_2\beta^n}{\beta-\alpha} = \frac{b_1\alpha^n - b_2\beta^n}{\alpha-\beta} \tag{1.7}$$

where

$$b_1 = W_1 - \beta W_0,\ b_2 = W_1 - \alpha W_0.$$

(1.7) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{1.8}$$

where

$$A_1 = \frac{W_1-\beta W_0}{\alpha-\beta},\ A_2 = \frac{W_1-\alpha W_0}{\beta-\alpha}.$$

Note that

$$\begin{aligned} A_1A_2 &= \frac{(W_1^2 - sW_0^2 - rW_1W_0)}{-(r^2+4s)}, \\ A_1 + A_2 &= W_0. \end{aligned}$$

We next find Binet's formula of generalized Fibonacci numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.3. (Binet's formula of generalized Fibonacci numbers)

$$W_n = \frac{d_1\alpha^n}{(\alpha-\beta)} + \frac{d_2\beta^n}{(\beta-\alpha)} \tag{1.9}$$

where

$$\begin{aligned} d_1 &= W_0\alpha + (W_1 - rW_0), \\ d_2 &= W_0\beta + (W_1 - rW_0)\beta. \end{aligned}$$

Proof. For a proof see [5, Theorem 1.2]. □
Note that from (1.7) and (1.9) we have

$$W_1 - \beta W_0 = W_0\alpha + (W_1 - rW_0), \tag{1.10}$$

$$W_1 - \alpha W_0 = W_0\beta + (W_1 - rW_0)\beta. \tag{1.11}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.7) or (1.9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha-\beta)} + \frac{\beta^n}{(\beta-\alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

1.2 Binet's Formula for the Single Root Case

In this subsection, we assume that the roots α and β of characteristic equation (1.5) are equal, i.e., $\alpha = \beta$. So (1.5) can be written as

$$x^2 - rx - s = (x-\alpha)^2 = x^2 - 2\alpha x + \alpha^2 = 0.$$

Note that in this case,

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ r &= 2\alpha, \\ s &= -\alpha^2 = -\frac{r^2}{4}, \\ r^2 + 4s &= 0. \end{aligned}$$

Using the root α and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.4 (Single Root Case)**.** Binet's formula of generalized Fibonacci numbers is

$$W_n = (D_1 + D_2 n)\alpha^n \tag{1.12}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}(W_1 - \alpha W_0). \end{aligned}$$

Proof. For a proof, see Soykan [36].

Note that (1.12) can be written as

$$W_n = \left(nW_1 - \frac{r}{2}(n-1)W_0\right)\left(\frac{r}{2}\right)^{n-1}.$$

We also see that

$$\begin{aligned} D_1 D_2 &= \frac{W_0(2W_1 - rW_0)}{r}, \\ D_1 + D_2 &= 2\frac{W_1}{r}. \end{aligned}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.7) or (1.9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2 The Sum Formula $\sum_{k=0}^{n} x^k W_{mk+j}^2$

In this section, we present sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers). The following theorem presents sum formulas of generalized (r,s) numbers (generalized Fibonacci numbers) in the case the roots α and β of characteristic equation (1.5) are distinct, i.e. $r^2 + 4s \neq 0$.

Theorem 2.1 (Distinct Roots Case)**.** Suppose that the roots α and β of characteristic equation (1.5) are distinct, i.e. $r^2+4s \neq 0$. Let x be a real (or complex) number. For all integers m and j, for generalized (r,s) numbers (generalized Fibonacci numbers), we have the following sum formulas:

(a) If $(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1) \neq 0$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{(r^2+4s)(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1)} \tag{2.1}$$

where
$\Omega_1 = (r^2+4s)((-s)^m x-1)((-s)^{2m} x - H_{2m})x^{n+1}W_{mn+j}^2 + (r^2+4s)(-s)^{2m}((-s)^m x - 1)x^{n+1}W_{mn-m+j}^2 + (r^2+4s)((-s)^m x-1)W_j^2 - (r^2+4s)(-s)^{2m}((-s)^m x - 1)xW_{j-m}^2 + 2(-s)^j (W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} x^n - 1)(H_{2m} - 2(-s)^m)x.$

(b) If $(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1) = u(x-a)(x-b)(x-c) = 0$ for some $u,a,b,c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x=a$ or $x=b$ or $x=c$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{(r^2+4s)(3(-s)^{3m}x^2 - 2(-s)^m(H_{2m}+(-s)^m)x + H_{2m} + (-s)^m)}$$

where
$\Omega_2 = (r^2+4s)((-s)^m((-s)^{2m}x - H_{2m})x^{n+1} + ((-s)^m x - 1)((-s)^{2m}(n+2)x - (n+1)H_{2m})x^n)W_{mn+j}^2 + (r^2+4s)(-s)^{2m}((-s)^m(n+2)x - (n+1))x^n W_{mn-m+j}^2 + (r^2+4s)(-s)^m W_j^2 - (r^2+4s)(-s)^{2m}(2(-s)^m x - 1)W_{j-m}^2 + 2(-s)^j (W_1^2 - sW_0^2 - rW_1W_0)(x^n(-s)^{mn}(n+1) - 1)(H_{2m} - 2(-s)^m).$

(c) If $(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1) = u(x-a)^2(x-c) = 0$ for some $u,a,c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x=c$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{(r^2+4s)(3(-s)^{3m}x^2 - 2(-s)^m(H_{2m}+(-s)^m)x + H_{2m} + (-s)^m)}$$

where
$\Omega_3 = (r^2+4s)((-s)^m((-s)^{2m}x - H_{2m})x^{n+1} + ((-s)^m x - 1)((-s)^{2m}(n+2)x - (n+1)H_{2m})x^n)W_{mn+j}^2 + (r^2+4s)(-s)^{2m}((-s)^m(n+2)x - (n+1))x^n W_{mn-m+j}^2 + (r^2+4s)(-s)^m W_j^2 - (r^2+4s)(-s)^{2m}(2(-s)^m x - 1)W_{j-m}^2 + 2(-s)^j (W_1^2 - sW_0^2 - rW_1W_0)(x^n(-s)^{mn}(n+1) - 1)(H_{2m} - 2(-s)^m)$
and if $x=a$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_4}{2(r^2+4s)(-s)^m(3(-s)^{2m}x - (-s)^m - H_{2m})}$$

where
$\Omega_4 = (r^2+4s)((-s)^{3m}(n+3)(n+2)x^2 - x(-s)^m(n+2)(n+1)(H_{2m}+(-s)^m) + n(n+1)H_{2m})x^{n-1}W_{mn+j}^2 + (r^2+4s)(-s)^{2m}(n+1)((2+n)(-s)^m x^n - nx^{n-1})W_{mn-m+j}^2 - 2(r^2+4s)(-s)^{3m}W_{j-m}^2 + 2n(n+1)(-s)^{mn+j}(W_1^2 - sW_0^2 - rW_1W_0)(H_{2m} - 2(-s)^m)x^{n-1}.$

(d) If $(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1) = u(x-a)^3 = 0$ for some $u,a \in \mathbb{C}$ with $u \neq 0$, i.e., $x=a$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_5}{6(-s)^{3m}(r^2+4s)}$$

where
$\Omega_5 = (r^2+4s)(n+1)((-s)^{3m}(n+3)(n+2)x^2 - n(-s)^m(n+2)(H_{2m}+(-s)^m)x + n(n-1)H_{2m})x^{n-2}W_{mn+j}^2 + n(-s)^{2m}(r^2+4s)(n+1)((n+2)(-s)^m x + 1 - n)x^{n-2}W_{mn-m+j}^2 + 2(n-1)n(n+1)(-s)^{mn+j}(H_{2m} - 2(-s)^m)(W_1^2 - sW_0^2 - rW_1W_0)x^{n-2}.$

Proof.

(a) *Note that*

$$\sum_{k=0}^{n-1} a^{mk+j} = a^j \left(\frac{(a^m)^n - 1}{a^m - 1} \right)$$

and

$$\begin{aligned}
\sum_{k=0}^{n} x^k W_{mk+j}^2 &= x^n W_{mn+j}^2 + \sum_{k=0}^{n-1} x^k W_{mk+j}^2 \\
&= x^n W_{mn+j}^2 + \sum_{k=0}^{n-1} (A_1 \alpha^{mk+j} + A_2 \beta^{mk+j})^2 x^k \\
&= x^n W_{mn+j}^2 + A_1^2 \alpha^{2j} \sum_{k=0}^{n-1} (\alpha^{2m} x)^k + A_2^2 \beta^{2j} \sum_{k=0}^{n-1} (\beta^{2m} x)^k + 2A_1 A_2 \alpha^j \beta^j \sum_{k=0}^{n-1} (\alpha^m \beta^m x)^k.
\end{aligned}$$

Simplifying the last equalities in the last two expression imply (2.1) as required.

(b) Note that we can write $(1 + (-s)^{2m} x^2 - x H_{2m})((-s)^m x - 1) = 0$ as

$$(-s)^{3m} (x^2 - x \frac{1}{(-s)^{2m}} H_{2m} + \frac{1}{(-s)^{2m}})(x - \frac{1}{(-s)^m}) = 0.$$

Solving this equation we find that

$$\begin{aligned}
x_1 &= a = \frac{1}{2(-s)^{2m}} \left(H_{2m} + \sqrt{H_{2m}^2 - 4(-s)^{2m}} \right), \\
x_2 &= b = \frac{1}{2(-s)^{2m}} \left(H_{2m} - \sqrt{H_{2m}^2 - 4(-s)^{2m}} \right), \\
x_3 &= c = \frac{1}{(-s)^m}.
\end{aligned}$$

If $H_{2m}^2 - 4(-s)^{2m} \neq 0$ then $a \neq b$. We assume that $b \neq c$. We use (2.1). For $x = a$, the right hand side of the above sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\sum_{k=0}^{n} a^k W_{mk+j}^2 = \left. \frac{\frac{d}{dx}(\Omega_1)}{\frac{d}{dx}((r^2 + 4s)(1 + (-s)^{2m} x^2 - x H_{2m})((-s)^m x - 1))} \right|_{x=a}.$$

The proof for the case $x = b$ and $x = c$ are the same.

(c) If $H_{2m}^2 - 4(-s)^{2m} = 0$ then $a = b = \dfrac{H_{2m}}{2(-s)^{2m}}$. We suppose that $a \neq c = \frac{1}{(-s)^m}$. If $x = c$ then the required result is obtained by (b). Now suppose that $x = a$. We use (2.1). For $x = a$, the right hand side of the above sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get the required result by using

$$\sum_{k=0}^{n} a^k W_{mk+j}^2 = \left. \frac{\frac{d^2}{dx^2}(\Omega_1)}{\frac{d^2}{dx^2}((r^2 + 4s)(1 + (-s)^{2m} x^2 - x H_{2m})((-s)^m x - 1))} \right|_{x=a}.$$

(d) If $H_{2m}^2 - 4(-s)^{2m} = 0$ then $a = b = \dfrac{H_{2m}}{2(-s)^{2m}}$. We suppose that $a = c = \frac{1}{(-s)^m}$. We use (2.1). For $x = a$, the right hand side of the above sum formula (2.1) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) by using

$$\sum_{k=0}^{n} a^k W_{mk+j}^2 = \left. \frac{\frac{d^3}{dx^3}(\Omega_1)}{\frac{d^3}{dx^3}((r^2 + 4s)(1 + (-s)^{2m} x^2 - x H_{2m})((-s)^m x - 1))} \right|_{x=a}$$

□

Note that (2.1) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{(r^2+4s)(1+(-s)^{2m}x^2 - xH_{2m})((-s)^m x - 1)}$$

where
$\Omega_6 = (r^2+4s)((-s)^m x-1)((-s)^{2m} x - H_{2m})x^{n+1}W_{mn+j}^2 + (r^2+4s)(-s)^{2m}((-s)^m x - 1)x^{n+1}W_{mn-m+j}^2 - (r^2+4s)((-s)^m x-1)((-s)^{2m} x - H_{2m})xW_j^2 - (r^2+4s)(-s)^{2m}((-s)^m x - 1)W_{j-m}^2 x + 2(-s)^j (W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} x^n - 1)(H_{2m} - 2(-s)^m)x.$

The following theorem presents sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers) in the case the roots α and β of characteristic equation (1.5) are equal, i.e., $\alpha = \beta$ so that $r^2 + 4s = 0$.

Theorem 2.2 (Single Root Case)**.** Assume that the roots α and β of characteristic equation (1.5) are equal, i.e., $\alpha = \beta$ so that $r^2 + 4s = 0$. Let x be a real (or complex) number. For all integers m and j, for generalized (r, s) numbers (generalized Fibonacci numbers), we have the following sum formulas:

(a) if $((-s)^m x - 1)^3 \neq 0$, i.e., $x \neq (-s)^{-m}$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_7}{((-s)^m x - 1)^3} \tag{2.2}$$

where
$\Omega_7 = ((-s)^m x-1)((-s)^{2m} x - 2(-s)^m)x^{n+1}W_{mn+j}^2 + (-s)^{2m}((-s)^m x - 1)x^{n+1}W_{mn-m+j}^2 + ((-s)^m x - 1)W_j^2 - (-s)^{2m}((-s)^m x - 1)xW_{j-m}^2 + 2m^2(-s)^{m+j-1}(W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} x^n - 1)x.$

(b) if $((-s)^m x - 1)^3 = 0$, i.e., $x = (-s)^{-m}$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \Omega_8$$

where
$\Omega_8 = \frac{1}{6s}(-s)^{-mn}(n+1)(3ns(-s)^m W_{mn-m+j}^2 - 3s(n-2)W_{mn+j}^2 - 2m^2 n(n-1)(-s)^{j+mn}(W_1^2 - sW_0^2 - rW_1W_0)).$

Proof. Note that

$$H_{2m} = \alpha^{2m} + \beta^{2m} = \left(\frac{r+\sqrt{r^2+4s}}{2}\right)^{2m} + \left(\frac{r-\sqrt{r^2+4s}}{2}\right)^{2m} \tag{2.3}$$

and if $r^2 + 4s = 0$ then $s = -\frac{r^2}{4} = -\alpha^2$ and

$$H_{2m} = 2\alpha^{2m} = 2(-s)^m.$$

By using (2.3), we see that

$$\lim_{s \to -\frac{r^2}{4}} \frac{(H_{2m} - 2(-s)^m)}{(r^2+4s)} = m^2(-s)^{m-1}. \tag{2.4}$$

(a) Use (2.4) and (2.1) (which is given in Theorem 2.1 (a)).

(b) We use (2.2). For $x = (-s)^{-m}$, the right hand side of the above sum formula (2.2) is an indeterminate form. Now, we can use L'Hospital rule (three times). Then we get (b) by using

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \left.\frac{\frac{d^3}{dx^3}(\Omega_7)}{\frac{d^3}{dx^3}\left(((-s)^m x - 1)^3\right)}\right|_{x=(-s)^{-m}} = \Omega_8.$$

□

Note that (2.2) can be written in the following form:

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_9}{((-s)^m x - 1)^3}$$

where
$\Omega_9 = ((-s)^m x - 1)((-s)^{2m} x - 2(-s)^m)x^{n+1} W_{mn+j}^2 + (-s)^{2m}((-s)^m x - 1)x^{n+1} W_{mn-m+j}^2 - x(-s)^m((-s)^m x - 2)((-s)^m x - 1)W_j^2 - (-s)^{2m}((-s)^m x - 1)xW_{j-m}^2 + 2m^2(-s)^{m+j-1}(W_1^2 - sW_0^2 - rW_1W_0)((-s)^{mn} x^n - 1)x.$

2.1 The Case $r = 1, s = 1$: Generalized Fibonacci Numbers

The following theorem presents sum formulas of generalized Fibonacci numbers (the case $r = 1, s = 1$).

Theorem 2.3. Let x be a real (or complex) number. For all integers m and j, for generalized Fibonacci numbers (the case $r = 1, s = 1$) we have the following sum formulas:

(a) If $(1 + (-1)^{2m}x^2 - xL_{2m})((-1)^m x - 1) \neq 0$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{5(1 + (-1)^{2m}x^2 - xL_{2m})((-1)^m x - 1)} \tag{2.5}$$

where
$\Omega_1 = 5((-1)^m x - 1)(x - L_{2m})x^{n+1} W_{mn+j}^2 + 5((-1)^m x - 1)x^{n+1} W_{mn-m+j}^2 + 5((-1)^m x - 1)W_j^2 - 5((-1)^m x - 1)xW_{j-m}^2 + 2(-1)^j(W_1^2 - W_0^2 - W_1W_0)((-1)^{mn} x^n - 1)(L_{2m} - 2(-1)^m)x.$

(b) If $(1 + (-1)^{2m}x^2 - xL_{2m})((-1)^m x - 1) = u(x-a)(x-b)(x-c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{5(3(-1)^{3m} x^2 - 2(-1)^m(L_{2m} + (-1)^m)x + L_{2m} + (-1)^m)}$$

where
$\Omega_2 = 5((-1)^m(x - L_{2m})x^{n+1} + ((-1)^m x - 1)((n+2)x - (n+1)L_{2m})x^n)W_{mn+j}^2 + 5((-1)^m(n+2)x - (n+1))x^n W_{mn-m+j}^2 + 5(-1)^m W_j^2 - 5(2(-1)^m x - 1)W_{j-m}^2 + 2(-1)^j(W_1^2 - W_0^2 - W_1W_0)(x^n(-1)^{mn}(n+1) - 1)(L_{2m} - 2(-1)^m).$

(c) If $(1 + (-1)^{2m}x^2 - xL_{2m})((-1)^m x - 1) = u(x-a)^2(x-c) = 0$ for some $u, a, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x = c$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{5(3(-1)^{3m} x^2 - 2(-1)^m(L_{2m} + (-1)^m)x + L_{2m} + (-1)^m)}$$

where

$\Omega_3 = 5((-1)^m(x-L_{2m})x^{n+1}+((-1)^m x-1)((n+2)x-(n+1)L_{2m})x^n)W^2_{mn+j}+5((-1)^m(n+2)x-(n+1))x^nW^2_{mn-m+j}+5(-1)^mW_j^2-5(2(-1)^mx-1)W^2_{j-m}+2(-1)^j(W_1^2-W_0^2-W_1W_0)(x^n(-1)^{mn}(n+1)-1)(L_{2m}-2(-1)^m)$
and if $x=a$ then

$$\sum_{k=0}^{n} x^kW^2_{mk+j} = \frac{\Omega_4}{10(-1)^m(3(-1)^{2m}x-(-1)^m-L_{2m})}$$

where
$\Omega_4 = 5((-1)^{3m}(n+3)(n+2)x^2-x(-1)^m(n+2)(n+1)(L_{2m}+(-1)^m)+n(n+1)L_{2m})x^{n-1}W^2_{mn+j}+5(n+1)((2+n)(-1)^mx^n-nx^{n-1})W^2_{mn-m+j}-10(-1)^{3m}W^2_{j-m}+2n(n+1)(-1)^{mn+j}(W_1^2-W_0^2-W_1W_0)(L_{2m}-2(-1)^m)x^{n-1}$.

(d) If $(1+(-1)^{2m}x^2-xL_{2m})((-1)^mx-1)=u(x-a)^3=0$ for some $u,a\in\mathbb{C}$ with $u\neq 0$, i.e., $x=a$, then

$$\sum_{k=0}^{n} x^kW^2_{mk+j} = \frac{\Omega_5}{30(-s)^{3m}}$$

where
$\Omega_5 = 5(n+1)((-1)^{3m}(n+3)(n+2)x^2-n(-1)^m(n+2)(L_{2m}+(-1)^m)x+n(n-1)L_{2m})x^{n-2}W^2_{mn+j}+5n(n+1)((n+2)(-1)^mx+1-n)x^{n-2}W^2_{mn-m+j}+2(n-1)n(n+1)(-1)^{mn+j}(L_{2m}-2(-1)^m)(W_1^2-W_0^2-W_1W_0)x^{n-2}$.

Proof. Take $r=1, s=1$ and $H_n=L_n$ in Theorem 2.1. □

Note that (2.5) can be written in the following form

$$\sum_{k=1}^{n} x^kW^2_{mk+j} = \frac{\Omega_6}{5(1+(-1)^{2m}x^2-xL_{2m})((-1)^mx-1)}$$

where
$\Omega_6 = 5((-1)^mx-1)(x-L_{2m})x^{n+1}W^2_{mn+j}+5((-1)^mx-1)x^{n+1}W^2_{mn-m+j}-5((-1)^mx-1)(x-L_{2m})xW_j^2-5((-1)^mx-1)W^2_{j-m}x+2(-1)^j(W_1^2-W_0^2-W_1W_0)((-1)^{mn}x^n-1)(L_{2m}-2(-1)^m)x$.

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.1. *For generalized Fibonacci numbers (the case $r=s=1$) we have the following sum formulas for $n\geq 0$:*

(a) $(m=1,\ j=0)$
If $(x+1)\left(x^2-3x+1\right)\neq 0$, *i.e.,* $x\neq -1$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^kW_k^2 = \frac{\Delta}{(x+1)(x^2-3x+1)}$$

where
$\Delta = (x+1)(x-3)x^{n+1}W_n^2+(x+1)x^{n+1}W^2_{n-1}+(x+1)W_0^2-x(x+1)(W_0-W_1)^2-2(W_1^2-W_0^2-W_1W_0)((-1)^nx^n-1)x$
and
if $(x+1)\left(x^2-3x+1\right)=0$, *i.e.,* $x=-1$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^kW_k^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where
$\Psi = (n(x+1)(x-3)+3x^2-4x-3)x^nW_n^2+(n+1+x(n+2))x^nW^2_{n-1}-(2x+1)(W_0-W_1)^2+W_0^2-2(W_1^2-W_0^2-W_1W_0)(x^n(-1)^n(n+1)-1)$.

(b) $(m = 2,\ j = 0)$
If $(x-1)\left(x^2-7x+1\right) \neq 0$, *i.e.*, $x \neq 1$, $x \neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x \neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(x-1)\left(x^2-7x+1\right)}$$

where
$\Delta = x^{n+1}(x-1)(x-7)W_{2n}^2 + (x-1)x^{n+1}W_{2n-2}^2 + (x-1)W_0^2 - (x-1)x(W_1-2W_0)^2 + 2(W_1^2-W_0^2-W_1W_0)(x^n-1)x$
and
if $(x-1)\left(x^2-7x+1\right) = 0$, *i.e.*, $x = 1$ *or* $x = \frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x = \frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where
$\Psi = (n(x-1)(x-7)+3x^2-16x+7)x^nW_{2n}^2 + ((n+2)x-(n+1))x^nW_{2n-2}^2 + W_0^2 - (2x-1)(W_1-2W_0)^2 + 2(W_1^2-W_0^2-W_1W_0)(x^n(n+1)-1)$.

(c) $(m = 2,\ j = 1)$
If $(x-1)\left(x^2-7x+1\right) \neq 0$, *i.e.*, $x \neq 1$, $x \neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x \neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{(x-1)(x^2-7x+1)}$$

where
$\Delta = (x-1)(x-7)x^{n+1}W_{2n+1}^2 + (x-1)x^{n+1}W_{2n-1}^2 + (x-1)W_1^2 - (x-1)x(W_0-W_1)^2 - 2(W_1^2-W_0^2-W_1W_0)(x^n-1)x$
and
if $(x-1)\left(x^2-7x+1\right) = 0$, *i.e.*, $x = 1$ *or* $x = \frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x = \frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where
$\Psi = (n(x-1)(x-7)+3x^2-16x+7)x^nW_{2n+1}^2 + ((n+2)x-(n+1))x^nW_{2n-1}^2 + W_1^2 - (2x-1)(W_0-W_1)^2 - 2(W_1^2-W_0^2-W_1W_0)(x^n(n+1)-1)$.

(d) $(m = -1,\ j = 0)$
If $(x+1)\left(x^2-3x+1\right) \neq 0$, *i.e.*, $x \neq -1$, $x \neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x \neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(x+1)\left(x^2-3x+1\right)}$$

where
$\Delta = (x+1)x^{n+1}W_{-n+1}^2 + (x+1)(x-3)x^{n+1}W_{-n}^2 + (x+1)W_0^2 - (x+1)xW_1^2 - 2(W_1^2-W_0^2-W_1W_0)((-1)^n x^n - 1)x$
if $(x+1)\left(x^2-3x+1\right) = 0$, *i.e.*, $x = -1$ *or* $x = \frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x = \frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where
$\Psi = (n(x+1)+2x+1)x^nW_{-n+1}^2 + (n(x+1)(x-3)+3x^2-4x-3)x^nW_{-n}^2 + W_0^2 - (2x+1)W_1^2 - 2(W_1^2-W_0^2-W_1W_0)(x^n(-1)^n(n+1)-1)$.

(e) $(m=-2,\ j=0)$
If $(x-1)\left(x^2-7x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{(x-1)(x^2-7x+1)}$$

where
$\Delta=(x-1)x^{n+1}W_{-2n+2}^2+(x-1)(x-7)x^{n+1}W_{-2n}^2+(x-1)W_0^2-(x-1)x(W_1+W_0)^2+2(W_1^2-W_0^2-W_1W_0)(x^n-1)x$
and
if $(x-1)\left(x^2-7x+1\right)=0$, *i.e.*, $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where
$\Psi=((n+2)x-(n+1))x^nW_{-2n+2}^2+(n(x-1)(x-7)+3x^2-16x+7)x^nW_{-2n}^2+W_0^2-(2x-1)(W_1+W_0)^2+2(W_1^2-W_0^2-W_1W_0)(x^n(n+1)-1)$.

(f) $(m=-2,\ j=1)$
If $(x-1)(x^2-7x+1)\neq 0$, *i.e.*, $x\neq 1$, $x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x-1)(x^2-7x+1)}$$

where
$\Delta=(x-1)x^{n+1}W_{-2n+3}^2+(x-1)(x-7)x^{n+1}W_{-2n+1}^2+(x-1)W_1^2-(x-1)x(W_0+2W_1)^2-2(W_1^2-W_0^2-W_1W_0)(x^n-1)x$
and
if $(x-1)(x^2-7x+1)=0$, *i.e.*, $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where
$\Psi=((n+2)x-(n+1))x^nW_{-2n+3}^2+(n(x-1)(x-7)+3x^2-16x+7)x^nW_{-2n+1}^2+W_1^2-(2x-1)(W_0+2W_1)^2-2(W_1^2-W_0^2-W_1W_0)(x^n(n+1)-1)$.

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0, F_1=1$).

Corollary 2.1. *For* $n\geq 0$, *Fibonacci numbers have the following properties:*

(a) $(m=1,\ j=0)$
If $(x+1)\left(x^2-3x+1\right)\neq 0$, *i.e.*, $x\neq -1$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_k^2 = \frac{(x+1)(x-3)x^{n+1}F_n^2+(x+1)x^{n+1}F_{n-1}^2-x(2(-1)^n x^n+x-1)}{(x+1)(x^2-3x+1)}$$

and

if $(x+1)\left(x^2-3x+1\right)=0$, *i.e.*, $x=-1$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_k^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where $\Psi=(n(x+1)(x-3)+3x^2-4x-3)x^nF_n^2+(n+1+x(n+2))x^nF_{n-1}^2-(2(-1)^n(n+1)x^n+2x-1)$.

(b) $(m=2,\ j=0)$

If $(x-1)\left(x^2-7x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{2k}^2 = \frac{x^{n+1}(x-1)(x-7)F_{2n}^2+(x-1)x^{n+1}F_{2n-2}^2+x(2x^n-x-1)}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)\left(x^2-7x+1\right)=0$, *i.e.*, $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=(n(x-1)(x-7)+3x^2-16x+7)x^nF_{2n}^2+((n+2)x-(n+1))x^nF_{2n-2}^2+2(n+1)x^n-2x-1$.

(c) $(m=2,\ j=1)$

If $(x-1)\left(x^2-7x+1\right)\neq 0$, *i.e.*, $x\neq 1$, $x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{2k+1}^2 = \frac{(x-1)(x-7)x^{n+1}F_{2n+1}^2+(x-1)x^{n+1}F_{2n-1}^2-(2x^{n+1}+x^2-4x+1)}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)\left(x^2-7x+1\right)=0$, *i.e.*, $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=(n(x-1)(x-7)+3x^2-16x+7)x^nF_{2n+1}^2+((n+2)x-(n+1))x^nF_{2n-1}^2-2((n+1)x^n+x-2)$.

(d) $(m=-1,\ j=0)$

If $(x+1)\left(x^2-3x+1\right)\neq 0$, *i.e.*, $x\neq -1$, $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-k}^2 = \frac{(x+1)x^{n+1}F_{-n+1}^2+(x+1)(x-3)x^{n+1}F_{-n}^2-x(2(-1)^nx^n+x-1)}{(x+1)(x^2-3x+1)}$$

if $(x+1)\left(x^2-3x+1\right)=0$, *i.e.*, $x=-1$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-k}^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where $\Psi=(n(x+1)+2x+1)x^nF_{-n+1}^2+(n(x+1)(x-3)+3x^2-4x-3)x^nF_{-n}^2-(2(-1)^n(n+1)x^n+2x-1)$.

(e) $(m = -2,\ j = 0)$

If $(x-1)\left(x^2-7x+1\right) \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x \neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k}^2 = \frac{(x-1)x^{n+1}F_{-2n+2}^2 + (x-1)(x-7)x^{n+1}F_{-2n}^2 + x(2x^n - x - 1)}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)\left(x^2-7x+1\right) = 0$, *i.e.,* $x = 1$ *or* $x = \frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x = \frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi = ((n+2)x-(n+1))x^n F_{-2n+2}^2 + (n(x-1)(x-7)+3x^2-16x+7)x^n F_{-2n}^2 + 2(n+1)x^n - 2x - 1.$

(f) $(m = -2,\ j = 1)$

If $(x-1)(x^2-7x+1) \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{7}{2}+\frac{3}{2}\sqrt{5}$, $x \neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1}^2 = \frac{(x-1)x^{n+1}F_{-2n+3}^2 + (x-1)(x-7)x^{n+1}F_{-2n+1}^2 - (2x^{n+1} + 4x^2 - 7x + 1)}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)(x^2-7x+1) = 0$, *i.e.,* $x = 1$ *or* $x = \frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x = \frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi = ((n+2)x-(n+1))x^n F_{-2n+3}^2 + (n(x-1)(x-7)+3x^2-16x+7)x^n F_{-2n+1}^2 - (2(n+1)x^n + 8x - 7).$

Taking $W_n = L_n$ with $L_0 = 2, L_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 2.2. *For* $n \geq 0$, *Lucas numbers have the following properties:*

(a) $(m = 1,\ j = 0)$

If $(x+1)\left(x^2-3x+1\right) \neq 0$, *i.e.,* $x \neq -1$, $x \neq \frac{3}{2}+\frac{1}{2}\sqrt{5}$, $x \neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_k^2 = \frac{(x+1)(x-3)x^{n+1}L_n^2 + (x+1)x^{n+1}L_{n-1}^2 - (-10\,(-1)^n\,x^{n+1} + x^2 + 7x - 4)}{(x+1)\,(x^2-3x+1)}$$

and

if $(x+1)\left(x^2-3x+1\right) = 0$, *i.e.,* $x = -1$ *or* $x = \frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x = \frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_k^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where $\Psi = (n(x+1)(x-3)+3x^2-4x-3)x^n L_n^2 + (n+1+x(n+2))x^n L_{n-1}^2 + 10\,(-1)^n\,(n+1)x^n - 2x - 7.$

(b) $(m=2,\ j=0)$
If $(x-1)\left(x^2-7x+1\right)\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5},\ x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5},$ *then*

$$\sum_{k=0}^{n} x^k L_{2k}^2 = \frac{x^{n+1}(x-1)(x-7)L_{2n}^2+(x-1)x^{n+1}L_{2n-2}^2-(10x^{n+1}+9x^2-23x+4)}{(x-1)(x^2-7x+1)}$$

and
if $(x-1)\left(x^2-7x+1\right)=0,$ *i.e.,* $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=(n(x-1)(x-7)+3x^2-16x+7)x^nL_{2n}^2+((n+2)x-(n+1))x^nL_{2n-2}^2-(10(n+1)x^n+18x-23).$

(c) $(m=2,\ j=1)$
If $(x-1)\left(x^2-7x+1\right)\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5},\ x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5},$ *then*

$$\sum_{k=0}^{n} x^k L_{2k+1}^2 = \frac{(x-1)(x-7)x^{n+1}L_{2n+1}^2+(x-1)x^{n+1}L_{2n-1}^2+10x^{n+1}-x^2-8x-1}{(x-1)(x^2-7x+1)}$$

and
if $(x-1)\left(x^2-7x+1\right)=0,$ *i.e.,* $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=(n(x-1)(x-7)+3x^2-16x+7)x^nL_{2n+1}^2+((n+2)x-(n+1))x^nL_{2n-1}^2+2(5(n+1)x^n-x-4).$

(d) $(m=-1,\ j=0)$
If $(x+1)\left(x^2-3x+1\right)\neq 0,$ *i.e.,* $x\neq -1,\ x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5},$ *then*

$$\sum_{k=0}^{n} x^k L_{-k}^2 = \frac{(x+1)x^{n+1}L_{-n+1}^2+(x+1)(x-3)x^{n+1}L_{-n}^2+10(-1)^n x^{n+1}-x^2-7x+4}{(x+1)(x^2-3x+1)}$$

and
if $(x+1)\left(x^2-3x+1\right)=0,$ *i.e.,* $x=-1$ *or* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-k}^2 = \frac{\Psi}{(3x^2-4x-2)}$$

where $\Psi=(n(x+1)+2x+1)x^nL_{-n+1}^2+(n(x+1)(x-3)+3x^2-4x-3)x^nL_{-n}^2+10(-1)^n(n+1)x^n-2x-7.$

(e) $(m=-2,\ j=0)$
If $(x-1)\left(x^2-7x+1\right)\neq 0,$ *i.e.,* $x\neq 1,\ x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5},\ x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5},$ *then*

$$\sum_{k=0}^{n} x^k L_{-2k}^2 = \frac{(x-1)x^{n+1}L_{-2n+2}^2+(x-1)(x-7)x^{n+1}L_{-2n}^2-(10x^{n+1}+9x^2-23x+4)}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)\left(x^2-7x+1\right)=0$, *i.e.,* $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-2k}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=((n+2)x-(n+1))x^nL_{-2n+2}^2+(n(x-1)(x-7)+3x^2-16x+7)x^nL_{-2n}^2-(10(n+1)x^n+18x-23)$.

(f) $(m=-2,\ j=1)$

If $(x-1)(x^2-7x+1)\neq 0$, *i.e.,* $x\neq 1,\ x\neq \frac{7}{2}+\frac{3}{2}\sqrt{5},\ x\neq \frac{7}{2}-\frac{3}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1}^2 = \frac{(x-1)x^{n+1}L_{-2n+3}^2+(x-1)(x-7)x^{n+1}L_{-2n+1}^2+10x^{n+1}-16x^2+7x-1}{(x-1)(x^2-7x+1)}$$

and

if $(x-1)(x^2-7x+1)=0$, *i.e.,* $x=1$ *or* $x=\frac{7}{2}+\frac{3}{2}\sqrt{5}$ *or* $x=\frac{7}{2}-\frac{3}{2}\sqrt{5}$ *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1}^2 = \frac{\Psi}{(3x^2-16x+8)}$$

where $\Psi=((n+2)x-(n+1))x^nL_{-2n+3}^2+(n(x-1)(x-7)+3x^2-16x+7)x^nL_{-2n+1}^2+10(n+1)x^n-32x+7$.

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 2.3. *For* $n\geq 0$, *Fibonacci numbers and Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} F_k^2 = 2F_n^2-F_{n-1}^2+(-1)^n$.
- **(b)** $\sum_{k=0}^{n} F_{2k}^2 = \frac{1}{5}(6F_{2n}^2-F_{2n-2}^2-2n+1)$.
- **(c)** $\sum_{k=0}^{n} F_{2k+1}^2 = \frac{1}{5}(6F_{2n+1}^2-F_{2n-1}^2+2n)$.
- **(d)** $\sum_{k=0}^{n} F_{-k}^2 = -F_{-n+1}^2+2F_{-n}^2+(-1)^n$.
- **(e)** $\sum_{k=0}^{n} F_{-2k}^2 = \frac{1}{5}(-F_{-2n+2}^2+6F_{-2n}^2-2n+1)$.
- **(f)** $\sum_{k=0}^{n} F_{-2k+1}^2 = \frac{1}{5}(-F_{-2n+3}^2+6F_{-2n+1}^2+2n+3)$.

2.

- **(a)** $\sum_{k=0}^{n} L_k^2 = 2L_n^2-L_{n-1}^2-5(-1)^n+2$.
- **(b)** $\sum_{k=0}^{n} L_{2k}^2 = \frac{1}{5}(6L_{2n}^2-L_{2n-2}^2+10n+5)$.
- **(c)** $\sum_{k=0}^{n} L_{2k+1}^2 = \frac{1}{5}(6L_{2n+1}^2-L_{2n-1}^2-10n)$.
- **(d)** $\sum_{k=0}^{n} L_{-k}^2 = -L_{-n+1}^2+2L_{-n}^2-5(-1)^n+2$.
- **(e)** $\sum_{k=0}^{n} L_{-2k}^2 = \frac{1}{5}(-L_{-2n+2}^2+6L_{-2n}^2+10n+5)$.
- **(f)** $\sum_{k=0}^{n} L_{-2k+1}^2 = \frac{1}{5}(-L_{-2n+3}^2+6L_{-2n+1}^2-10n+15)$.

2.2 The Case $r = 2, s = 1$: Generalized Pell Numbers

The following theorem presents sum formulas of generalized Pell numbers (the case $r = 2, s = 1$).

Theorem 2.4. Let x be a real (or complex) number. For all integers m and j, for generalized Pell numbers (the case $r = 2, s = 1$) we have the following sum formulas:

(a) If $(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1) \neq 0$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{8(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1)} \tag{2.6}$$

where
$\Omega_1 = 8((-1)^m x-1)((-1)^{2m} x-Q_{2m})x^{n+1}W_{mn+j}^2+8(-1)^{2m}((-1)^m x-1)x^{n+1}W_{mn-m+j}^2+8((-1)^m x-1)W_j^2-8(-1)^{2m}((-1)^m x-1)xW_{j-m}^2+2(-1)^j(W_1^2-W_0^2-2W_1W_0)((-1)^{mn}x^n-1)(Q_{2m}-2(-1)^m)x.$

(b) If $(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1) = u(x-a)(x-b)(x-c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{8(3(-1)^{3m}x^2 - 2(-1)^m(Q_{2m}+(-1)^m)x + Q_{2m} + (-1)^m)}$$

where
$\Omega_2 = 8((-1)^m((-1)^{2m}x-Q_{2m})x^{n+1}+((-1)^m x-1)((-1)^{2m}(n+2)x-(n+1)Q_{2m})x^n)W_{mn+j}^2+8(-1)^{2m}((-1)^m(n+2)x-(n+1))x^nW_{mn-m+j}^2+8(-1)^mW_j^2-8(-1)^{2m}(2(-1)^m x-1)W_{j-m}^2+2(-1)^j(W_1^2-W_0^2-2W_1W_0)(x^n(-1)^{mn}(n+1)-1)(Q_{2m}-2(-1)^m).$

(c) If $(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1) = u(x-a)^2(x-c) = 0$ for some $u, a, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x = c$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{8(3(-1)^{3m}x^2 - 2(-1)^m(Q_{2m}+(-1)^m)x + Q_{2m} + (-1)^m)}$$

where
$\Omega_3 = 8((-1)^m((-1)^{2m}x-Q_{2m})x^{n+1}+((-1)^m x-1)((-1)^{2m}(n+2)x-(n+1)Q_{2m})x^n)W_{mn+j}^2+8(-1)^{2m}((-1)^m(n+2)x-(n+1))x^nW_{mn-m+j}^2+8(-1)^mW_j^2-8(-1)^{2m}(2(-1)^m x-1)W_{j-m}^2+2(-1)^j(W_1^2-W_0^2-2W_1W_0)(x^n(-1)^{mn}(n+1)-1)(Q_{2m}-2(-1)^m)$
and if $x = a$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_4}{16(-1)^m(3(-1)^{2m}x - (-1)^m - Q_{2m})}$$

where
$\Omega_4 = 8((-1)^{3m}(n+3)(n+2)x^2-x(-1)^m(n+2)(n+1)(Q_{2m}+(-1)^m)+n(n+1)Q_{2m})x^{n-1}W_{mn+j}^2+8(-1)^{2m}(n+1)((2+n)(-1)^m x^n-nx^{n-1})W_{mn-m+j}^2-16(-1)^{3m}W_{j-m}^2+2n(n+1)(-1)^{mn+j}(W_1^2-W_0^2-2W_1W_0)(Q_{2m}-2(-1)^m)x^{n-1}.$

(d) If $(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1) = u(x-a)^3 = 0$ for some $u, a \in \mathbb{C}$ with $u \neq 0$, i.e., $x = a$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_5}{48(-1)^{3m}}$$

where
$\Omega_5 = 8(n+1)((-1)^{3m}(n+3)(n+2)x^2-n(-1)^m(n+2)(Q_{2m}+(-1)^m)x+n(n-1)Q_{2m})x^{n-2}W_{mn+j}^2+n(-1)^{2m}8(n+1)((n+2)(-1)^m x+1-n)x^{n-2}W_{mn-m+j}^2+2(n-1)n(n+1)(-1)^{mn+j}(Q_{2m}-2(-1)^m)(W_1^2-W_0^2-2W_1W_0)x^{n-2}.$

Proof. *Take* $r = 2, s = 1$ *and* $H_n = Q_n$ in Theorem 2.1. □

Note that (2.6) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{8(1+(-1)^{2m}x^2 - xQ_{2m})((-1)^m x - 1)}$$

where

$\Omega_6 = 8((-1)^m x - 1)((-1)^{2m} x - Q_{2m})x^{n+1}W_{mn+j}^2 + 8(-1)^{2m}((-1)^m x - 1)x^{n+1}W_{mn-m+j}^2 - 8((-1)^m x - 1)((-1)^{2m} x - Q_{2m})xW_j^2 - 8(-1)^{2m}((-1)^m x - 1)W_{j-m}^2 x + 2(-1)^j(W_1^2 - W_0^2 - 2W_1W_0)((-1)^{mn} x^n - 1)(Q_{2m} - 2(-1)^m)x$.

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.2. *For generalized Pell numbers (the case* $r = 2, s = 1$*) we have the following sum formulas for* $n \geq 0$*:*

(a) $(m = 1, j = 0)$

If $(x+1)(x^2 - 6x + 1) \neq 0$*, i.e.,* $x \neq -1$, $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$*, then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{-(x+1)(x^2 - 6x + 1)}$$

where

$\Delta = -(x+1)(x-6)x^{n+1}W_n^2 - (x+1)x^{n+1}W_{n-1}^2 - (x+1)W_0^2 + (x+1)x(W_1 - 2W_0)^2 + 2(W_1^2 - W_0^2 - 2W_1W_0)((-1)^n x^n - 1)x$

and

if $(x+1)(x^2 - 6x + 1)$*, i.e.,* $x = -1$ *or* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{(-3x^2 + 10x + 5)}$$

where

$\Psi = -(n(x+1)(x-6) + 3x^2 - 10x - 6)x^n W_n^2 - ((n+2)x + (n+1))x^n W_{n-1}^2 - W_0^2 - (-2x - 1)(W_1 - 2W_0)^2 + 2(W_1^2 - W_0^2 - 2W_1W_0)(x^n(-1)^n(n+1) - 1)$.

(b) $(m = 2, j = 0)$

If $(x-1)(x^2 - 34x + 1) \neq 0$*, i.e.,* $x \neq 1$, $x \neq 17 + 12\sqrt{2}$, $x \neq 17 - 12\sqrt{2}$*, then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{-(x-1)(x^2 - 34x + 1)}$$

where

$\Delta = -(x-1)(x-34)x^{n+1}W_{2n}^2 - (x-1)x^{n+1}W_{2n-2}^2 - (x-1)W_0^2 + (x-1)x(2W_1 - 5W_0)^2 - 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n - 1)x$

and

if $(x-1)(x^2 - 34x + 1) = 0$*, i.e.,* $x = 1$ *or* $x = 17 + 12\sqrt{2}$ *or* $x = 17 - 12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{(3x^2 - 70x + 35)}$$

where

$\Psi = (n(x-1)(x-34) + 3x^2 - 70x + 34)x^n W_{2n}^2 + ((n+2)x - (n+1))x^n W_{2n-2}^2 + W_0^2 - (2x - 1)(2W_1 - 5W_0)^2 + 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n(n+1) - 1)$.

(c) $(m = 2,\ j = 1)$

If $(x-1)(x^2-34x+1) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 17+12\sqrt{2},\ x \neq 17-12\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{-(x-1)(x^2-34x+1)}$$

where

$\Delta = -(x-1)(x-34)x^{n+1}W_{2n+1}^2 - (x-1)x^{n+1}W_{2n-1}^2 - (x-1)W_1^2 + (x-1)x(W_1-2W_0)^2 + 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n-1)x$

and

if $(x-1)(x^2-34x+1) = 0,$ *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where

$\Psi = (n(x-1)(x-34) + 3x^2 - 70x + 34)x^nW_{2n+1}^2 + ((n+2)x-(n+1))x^nW_{2n-1}^2 + W_1^2 - (2x-1)(W_1-2W_0)^2 - 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n(n+1)-1).$

(d) $(m = -1,\ j = 0)$

If $(x+1)(x^2-6x+1) \neq 0,$ *i.e.,* $x \neq -1,\ x \neq 3+2\sqrt{2},\ x \neq 3-2\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{-(x+1)(x^2-6x+1)}$$

where

$\Delta = -(x+1)x^{n+1}W_{-n+1}^2 - (x+1)(x-6)x^{n+1}W_{-n}^2 - (x+1)W_0^2 + (x+1)xW_1^2 + 2(W_1^2 - W_0^2 - 2W_1W_0)((-1)^n x^n - 1)x$

and

if $(x+1)(x^2-6x+1) = 0,$ *i.e.,* $x = -1$ *or* $x = 3+2\sqrt{2}$ *or* $x = 3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{(-3x^2+10x+5)}$$

where

$\Psi = -((n+2)x+(n+1))x^nW_{-n+1}^2 - (n(x+1)(x-6) + 3x^2 - 10x - 6)x^nW_{-n}^2 - W_0^2 - (-2x-1)W_1^2 + 2(W_1^2 - W_0^2 - 2W_1W_0)(x^n(-1)^n(n+1)-1).$

(e) $(m = -2,\ j = 0)$

If $(x-1)(x^2-34x+1) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 17+12\sqrt{2},\ x \neq 17-12\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{-(x-1)(x^2-34x+1)}$$

where

$\Delta = -(x-1)x^{n+1}W_{-2n+2}^2 - (x-1)(x-34)x^{n+1}W_{-2n}^2 - (x-1)W_0^2 + (x-1)xW_2^2 - 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n-1)x$

and

if $(x-1)(x^2-34x+1) = 0,$ *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where

$\Psi = ((n+2)x-(n+1))x^nW_{-2n+2}^2 + (n(x-1)(x-34) + 3x^2 - 70x + 34)x^nW_{-2n}^2 + W_0^2 - (2x-1)W_2^2 + 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n(n+1)-1).$

(f) $(m = -2,\ j = 1)$

If $(x-1)(x^2-34x+1) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17+12\sqrt{2}$, $x \neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{-(x-1)(x^2-34x+1)}$$

where

$\Delta = -(x-1)x^{n+1}W_{-2n+3}^2 - (x-1)(x-34)x^{n+1}W_{-2n+1}^2 - (x-1)W_1^2 + (x-1)xW_3^2 + 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n-1)x$

and

if $(x-1)(x^2-34x+1) = 0$, *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where

$\Psi = ((n+2)x-(n+1))x^nW_{-2n+3}^2 + (n(x-1)(x-34) - 70x + 3x^2 + 34)x^nW_{-2n+1}^2 + W_1^2 - (2x-1)W_3^2 - 8(W_1^2 - W_0^2 - 2W_1W_0)(x^n(n+1)-1)$.

From the above proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 2.4. *For* $n \geq 0$, *Pell numbers have the following properties:*

(a) $(m = 1,\ j = 0)$

If $(x+1)(x^2-6x+1) \neq 0$, *i.e.,* $x \neq -1$, $x \neq 3+2\sqrt{2}$, $x \neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_k^2 = \frac{-(x+1)(x-6)x^{n+1}P_n^2 - (x+1)x^{n+1}P_{n-1}^2 + x(2(-1)^n x^n + x - 1)}{-(x+1)(x^2-6x+1)}$$

and

if $(x+1)(x^2-6x+1)$, *i.e.,* $x = -1$ *or* $x = 3+2\sqrt{2}$ *or* $x = 3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_k^2 = \frac{\Psi}{(-3x^2+10x+5)}$$

where $\Psi = -(n(x+1)(x-6) + 3x^2 - 10x - 6)x^nP_n^2 - ((n+2)x + (n+1))x^nP_{n-1}^2 + 2(-1)^n(n+1)x^n + 2x - 1$.

(b) $(m = 2,\ j = 0)$

If $(x-1)(x^2-34x+1) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 17+12\sqrt{2}$, $x \neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k}^2 = \frac{-(x-1)(x-34)x^{n+1}P_{2n}^2 - (x-1)x^{n+1}P_{2n-2}^2 + 4x(-2x^n + x + 1)}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1) = 0$, *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{2k}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = (n(x-1)(x-34) + 3x^2 - 70x + 34)x^nP_{2n}^2 + ((n+2)x - (n+1))x^nP_{2n-2}^2 + 8(n+1)x^n - 8x - 4$.

(c) $(m = 2,\ j = 1)$

If $(x-1)(x^2-34x+1) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 17+12\sqrt{2},\ x \neq 17-12\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k P_{2k+1}^2 = \frac{-(x-1)(x-34)x^{n+1}P_{2n+1}^2 - (x-1)x^{n+1}P_{2n-1}^2 + 8x^{n+1} + x^2 - 10x + 1}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1) = 0,$ *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = (n(x-1)(x-34) + 3x^2 - 70x + 34)x^n P_{2n+1}^2 + ((n+2)x - (n+1))x^n P_{2n-1}^2 - 2(4(n+1)x^n + x - 5).$

(d) $(m = -1,\ j = 0)$

If $(x+1)(x^2-6x+1) \neq 0,$ *i.e.,* $x \neq -1,\ x \neq 3+2\sqrt{2},\ x \neq 3-2\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k P_{-k}^2 = \frac{-(x+1)x^{n+1}P_{-n+1}^2 - (x+1)(x-6)x^{n+1}P_{-n}^2 + x(2(-1)^n x^n + x - 1)}{-(x+1)(x^2-6x+1)}$$

and

if $(x+1)(x^2-6x+1) = 0,$ *i.e.,* $x = -1$ *or* $x = 3+2\sqrt{2}$ *or* $x = 3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-k}^2 = \frac{\Psi}{(-3x^2+10x+5)}$$

where $\Psi = -((n+2)x + (n+1))x^n P_{-n+1}^2 - (n(x+1)(x-6) + 3x^2 - 10x - 6)x^n P_{-n}^2 + 2(-1)^n (n+1)x^n + 2x - 1.$

(e) $(m = -2,\ j = 0)$

If $(x-1)(x^2-34x+1) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 17+12\sqrt{2},\ x \neq 17-12\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k}^2 = \frac{-(x-1)x^{n+1}P_{-2n+2}^2 - (x-1)(x-34)x^{n+1}P_{-2n}^2 + 4x(-2x^n + x + 1)}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1) = 0,$ *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = ((n+2)x - (n+1))x^n P_{-2n+2}^2 + (n(x-1)(x-34) + 3x^2 - 70x + 34)x^n P_{-2n}^2 + 4(2(n+1)x^n - 2x - 1).$

(f) $(m = -2,\ j = 1)$

If $(x-1)(x^2-34x+1) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 17+12\sqrt{2},\ x \neq 17-12\sqrt{2},$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1}^2 = \frac{-(x-1)x^{n+1}P_{-2n+3}^2 - (x-1)(x-34)x^{n+1}P_{-2n+1}^2 + 8x^{n+1} + 25x^2 - 34x + 1}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1) = 0,$ *i.e.,* $x = 1$ *or* $x = 17+12\sqrt{2}$ *or* $x = 17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = ((n+2)x - (n+1))x^n P_{-2n+3}^2 + (n(x-1)(x-34) - 70x + 3x^2 + 34)x^n P_{-2n+1}^2 - 2(4(n+1)x^n + 25x - 17).$

Taking $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 2.5. *For $n \geq 0$, Pell-Lucas numbers have the following properties:*

(a) $(m = 1,\ j = 0)$
If $(x+1)(x^2-6x+1) \neq 0$, i.e., $x \neq -1$, $x \neq 3+2\sqrt{2}$, $x \neq 3-2\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_k^2 = \frac{-(x+1)(x-6)x^{n+1}Q_n^2 - (x+1)x^{n+1}Q_{n-1}^2 + 4(-4(-1)^n x^{n+1} + x^2 + 4x - 1)}{-(x+1)(x^2-6x+1)}$$

and
if $(x+1)(x^2-6x+1)$, i.e., $x = -1$ or $x = 3+2\sqrt{2}$ or $x = 3-2\sqrt{2}$ then

$$\sum_{k=0}^{n} x^k Q_k^2 = \frac{\Psi}{(-3x^2+10x+5)}$$

where $\Psi = -(n(x+1)(x-6)+3x^2-10x-6)x^nQ_n^2 - ((n+2)x+(n+1))x^nQ_{n-1}^2 - 8(2(-1)^n(n+1)x^n - x - 2)$.

(b) $(m = 2,\ j = 0)$
If $(x-1)(x^2-34x+1) \neq 0$, i.e., $x \neq 1$, $x \neq 17+12\sqrt{2}$, $x \neq 17-12\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_{2k}^2 = \frac{-(x-1)(x-34)x^{n+1}Q_{2n}^2 - (x-1)x^{n+1}Q_{2n-2}^2 + 4(16x^{n+1} + 9x^2 - 26x + 1)}{-(x-1)(x^2-34x+1)}$$

and
if $(x-1)(x^2-34x+1) = 0$, i.e., $x = 1$ or $x = 17+12\sqrt{2}$ or $x = 17-12\sqrt{2}$ then

$$\sum_{k=0}^{n} x^k Q_{2k}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = (n(x-1)(x-34)+3x^2-70x+34)x^nQ_{2n}^2 + ((n+2)x-(n+1))x^nQ_{2n-2}^2 - 8(8(n+1)x^n + 9x - 13)$.

(c) $(m = 2,\ j = 1)$
If $(x-1)(x^2-34x+1) \neq 0$, i.e., $x \neq 1$, $x \neq 17+12\sqrt{2}$, $x \neq 17-12\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_{2k+1}^2 = \frac{-(x-1)(x-34)x^{n+1}Q_{2n+1}^2 - (x-1)x^{n+1}Q_{2n-1}^2 + 4(-16x^{n+1} + x^2 + 14x + 1)}{-(x-1)(x^2-34x+1)}$$

and
if $(x-1)(x^2-34x+1) = 0$, i.e., $x = 1$ or $x = 17+12\sqrt{2}$ or $x = 17-12\sqrt{2}$ then

$$\sum_{k=0}^{n} x^k Q_{2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = (n(x-1)(x-34)+3x^2-70x+34)x^nQ_{2n+1}^2 + ((n+2)x-(n+1))x^nQ_{2n-1}^2 + 8(8(n+1)x^n - x - 7)$.

(d) $(m = -1,\ j = 0)$
If $(x+1)(x^2-6x+1) \neq 0$, i.e., $x \neq -1$, $x \neq 3+2\sqrt{2}$, $x \neq 3-2\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_{-k}^2 = \frac{-(x+1)x^{n+1}Q_{-n+1}^2 - (x+1)(x-6)x^{n+1}Q_{-n}^2 + 4((-4(-1)^n x^{n+1} + x^2 + 4x - 1)}{-(x+1)(x^2-6x+1)}$$

and

if $(x+1)(x^2-6x+1)=0$, *i.e.,* $x=-1$ *or* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-k}^2 = \frac{\Psi}{(-3x^2+10x+5)}$$

where $\Psi = -((n+2)x+(n+1))x^n Q_{-n+1}^2 - (n(x+1)(x-6)+3x^2-10x-6)x^n Q_{-n}^2 - 8(2(-1)^n(n+1)x^n - x - 2)$.

(e) $(m=-2,\ j=0)$

If $(x-1)(x^2-34x+1)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k}^2 = \frac{-(x-1)x^{n+1}Q_{-2n+2}^2-(x-1)(x-34)x^{n+1}Q_{-2n}^2+4(16x^{n+1}+9x^2-26x+1)}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1)=0$, *i.e.,* $x=1$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-2k}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = ((n+2)x-(n+1))x^n Q_{-2n+2}^2 + (n(x-1)(x-34)+3x^2-70x+34)x^n Q_{-2n}^2 - 8(8(n+1)x^n+9x-13)$.

(f) $(m=-2,\ j=1)$

If $(x-1)(x^2-34x+1)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 17+12\sqrt{2}$, $x\neq 17-12\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k+1}^2 = \frac{-(x-1)x^{n+1}Q_{-2n+3}^2-(x-1)(x-34)x^{n+1}Q_{-2n+1}^2+4(-16x^{n+1}+49x^2-34x+1)}{-(x-1)(x^2-34x+1)}$$

and

if $(x-1)(x^2-34x+1)=0$, *i.e.,* $x=1$ *or* $x=17+12\sqrt{2}$ *or* $x=17-12\sqrt{2}$ *then*

$$\sum_{k=0}^{n} x^k Q_{-2k+1}^2 = \frac{\Psi}{(3x^2-70x+35)}$$

where $\Psi = ((n+2)x-(n+1))x^n Q_{-2n+3}^2 + (n(x-1)(x-34)-70x+3x^2+34)x^n Q_{-2n+1}^2 + 8(8(n+1)x^n-49x+17)$.

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 2.6. *For* $n\geq 0$, *Pell numbers and Pell-Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} P_k^2 = \frac{1}{4}(5P_n^2 - P_{n-1}^2 + (-1)^n)$.
- **(b)** $\sum_{k=0}^{n} P_{2k}^2 = \frac{1}{32}(33P_{2n}^2 - P_{2n-2}^2 - 8n + 4)$.
- **(c)** $\sum_{k=0}^{n} P_{2k+1}^2 = \frac{1}{32}(33P_{2n+1}^2 - P_{2n-1}^2 + 8n)$.
- **(d)** $\sum_{k=0}^{n} P_{-k}^2 = \frac{1}{4}(5P_{-n}^2 - P_{-n+1}^2 + (-1)^n)$.
- **(e)** $\sum_{k=0}^{n} P_{-2k}^2 = \frac{1}{32}(-P_{-2n+2}^2 + 33P_{-2n}^2 - 8n + 4)$.
- **(f)** $\sum_{k=0}^{n} P_{-2k+1}^2 = \frac{1}{32}(-P_{-2n+3}^2 + 33P_{-2n+1}^2 + 8n + 24)$.

2.

(a) $\sum_{k=0}^{n} Q_k^2 = \frac{1}{4}(5Q_n^2 - Q_{n-1}^2 - 8(-1)^n + 8)$.
(b) $\sum_{k=0}^{n} Q_{2k}^2 = \frac{1}{32}(33Q_{2n}^2 - Q_{2n-2}^2 + 64n + 32)$.
(c) $\sum_{k=0}^{n} Q_{2k+1}^2 = \frac{1}{32}(33Q_{2n+1}^2 - Q_{2n-1}^2 - 64n)$.
(d) $\sum_{k=0}^{n} Q_{-k}^2 = \frac{1}{4}(-Q_{-n+1}^2 + 5Q_{-n}^2 - 8(-1)^n + 8)$.
(e) $\sum_{k=0}^{n} Q_{-2k}^2 = \frac{1}{32}(-Q_{-2n+2}^2 + 33Q_{-2n}^2 + 64n + 32)$.
(f) $\sum_{k=0}^{n} Q_{-2k+1}^2 = \frac{1}{32}(-Q_{-2n+3}^2 + 33Q_{-2n+1}^2 - 64n + 192)$.

2.3 The Case $r = 1, s = 2$: Generalized Jacobsthal Numbers

The following theorem presents sum formulas of generalized Jacobsthal numbers (the case $r = 1, s = 2$).

Theorem 2.5. Let x be a real (or complex) number. For all integers m and j, for generalized Jacobsthal numbers (the case $r = 1, s = 2$) we have the following sum formulas:

(a) If $(1 + (-2)^{2m}x^2 - xj_{2m})((-2)^m x - 1) \neq 0$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_1}{9(1 + (-2)^{2m}x^2 - xj_{2m})((-2)^m x - 1)} \tag{2.7}$$

where
$\Omega_1 = 9((-2)^m x - 1)((-2)^{2m} x - j_{2m})x^{n+1}W_{mn+j}^2 + 9(-2)^{2m}((-2)^m x - 1)x^{n+1}W_{mn-m+j}^2 + 9((-2)^m x - 1)W_j^2 - 9(-2)^{2m}((-2)^m x - 1)xW_{j-m}^2 + 2(-2)^j (W_1^2 - 2W_0^2 - W_1W_0)((-2)^{mn} x^n - 1)(j_{2m} - 2(-2)^m)x$.

(b) If $(1 + (-2)^{2m}x^2 - xj_{2m})((-2)^m x - 1) = u(x - a)(x - b)(x - c) = 0$ for some $u, a, b, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq b \neq c$, i.e., $x = a$ or $x = b$ or $x = c$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_2}{9(3(-2)^{3m} x^2 - 2(-2)^m (j_{2m} + (-2)^m)x + j_{2m} + (-2)^m)}$$

where
$\Omega_2 = 9((-2)^m ((-2)^{2m} x - j_{2m})x^{n+1} + ((-2)^m x - 1)((-2)^{2m}(n+2)x - (n+1)j_{2m})x^n)W_{mn+j}^2 + 9(-2)^{2m}((-2)^m(n+2)x - (n+1))x^n W_{mn-m+j}^2 + 9(-2)^m W_j^2 - 9(-2)^{2m}(2(-2)^m x - 1)W_{j-m}^2 + 2(-2)^j (W_1^2 - 2W_0^2 - W_1W_0)(x^n(-2)^{mn}(n+1) - 1)(j_{2m} - 2(-2)^m)$.

(c) If $(1 + (-2)^{2m}x^2 - xj_{2m})((-2)^m x - 1) = u(x - a)^2(x - c) = 0$ for some $u, a, c \in \mathbb{C}$ with $u \neq 0$ and $a \neq c$ then if $x = c$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_3}{9(3(-2)^{3m} x^2 - 2(-2)^m (j_{2m} + (-2)^m)x + j_{2m} + (-2)^m)}$$

where
$\Omega_3 = 9((-2)^m ((-2)^{2m} x - j_{2m})x^{n+1} + ((-2)^m x - 1)((-2)^{2m}(n+2)x - (n+1)j_{2m})x^n)W_{mn+j}^2 + 9(-2)^{2m}((-2)^m(n+2)x - (n+1))x^n W_{mn-m+j}^2 + 9(-2)^m W_j^2 - 9(-2)^{2m}(2(-2)^m x - 1)W_{j-m}^2 + 2(-2)^j (W_1^2 - 2W_0^2 - W_1W_0)(x^n(-2)^{mn}(n+1) - 1)(j_{2m} - 2(-2)^m)$
and if $x = a$ then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_4}{18(-2)^m (3(-2)^{2m} x - (-2)^m - j_{2m})}$$

where
$\Omega_4 = 9((-2)^{3m}(n+3)(n+2)x^2 - x(-2)^m(n+2)(n+1)(j_{2m} + (-2)^m) + n(n+1)j_{2m})x^{n-1}W_{mn+j}^2 + 9(-2)^{2m}(n+1)((2+n)(-2)^m x^n - nx^{n-1})W_{mn-m+j}^2 - 18(-2)^{3m} W_{j-m}^2 + 2n(n+1)(-2)^{mn+j} (W_1^2 - 2W_0^2 - W_1W_0)(j_{2m} - 2(-2)^m)x^{n-1}$.

(d) If $(1+(-2)^{2m}x^2 - xj_{2m})((-2)^m x-1) = u(x-a)^3 = 0$ for some $u,a \in \mathbb{C}$ with $u \neq 0$, i.e., $x=a$, then

$$\sum_{k=0}^{n} x^k W_{mk+j}^2 = \frac{\Omega_5}{54\,(-2)^{3m}}$$

where
$\Omega_5 = 9\,(n+1)\,((-2)^{3m}\,(n+3)(n+2)x^2 - n\,(-2)^m\,(n+2)(j_{2m}+(-2)^m)x + n(n-1)j_{2m})x^{n-2}W_{mn+j}^2 + 9n\,(-2)^{2m}\,(n+1)\,((n+2)\,(-2)^m\,x+1-n)\,x^{n-2}W_{mn-m+j}^2 + 2(n-1)n(n+1)\,(-2)^{mn+j}\,(j_{2m} - 2\,(-2)^m)(W_1^2-2W_0^2-W_1W_0)x^{n-2}$.

Proof. *Take* $r=1, s=2$ and $H_n = j_n$ in Theorem 2.1. □

Note that (2.7) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j}^2 = \frac{\Omega_6}{9(1+(-2)^{2m}x^2 - xj_{2m})((-2)^m x-1)}$$

where
$\Omega_6 = 9((-2)^m\,x-1)((-2)^{2m}\,x - j_{2m})x^{n+1}W_{mn+j}^2 + 9\,(-2)^{2m}\,((-2)^m x-1)x^{n+1}W_{mn-m+j}^2 - 9((-2)^m\,x-1)((-2)^{2m}\,x - j_{2m})xW_j^2 - 9\,(-2)^{2m}\,((-2)^m x-1)W_{j-m}^2 x + 2\,(-2)^j\,(W_1^2-2W_0^2 - W_1W_0)((-2)^{mn}\,x^n-1)(j_{2m}-2\,(-2)^m)x$.

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 2.3. *For generalized Jacobsthal numbers (the case* $r=1, s=2$*) we have the following sum formulas for* $n \geq 0$*:*

(a) $(m=1,\ j=0)$

If $(2x+1)\,(4x-1)\,(x-1) \neq 0$, *i.e.,* $x \neq -\frac{1}{2}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Delta}{(2x+1)\,(4x-1)\,(x-1)}$$

where
$\Delta = (2x+1)(4x-5)x^{n+1}W_n^2 + 4(2x+1)x^{n+1}W_{n-1}^2 + (2x+1)W_0^2 - (2x+1)x(W_0-W_1)^2 - 2(W_1^2-2W_0^2-W_1W_0)((-2)^n\,x^n-1)x$

and

if $(2x+1)\,(4x-1)\,(x-1) = 0$, *i.e.,* $x=-\frac{1}{2}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_k^2 = \frac{\Psi}{3\,(8x^2-4x-1)}$$

where
$\Psi = (n(2x+1)(4x-5)+24x^2-12x-5)x^nW_n^2 + 4(2(n+2)x+(n+1))x^nW_{n-1}^2 + 2W_0^2 - (4x+1)(W_0-W_1)^2 - 2(W_1^2-2W_0^2-W_1W_0)((-2)^n\,(n+1)x^n-1)$.

(b) $(m=2,\ j=0)$

If $(4x-1)\,(16x-1)\,(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Delta}{(4x-1)\,(16x-1)\,(x-1)}$$

where
$\Delta = (4x-1)(16x-17)x^{n+1}W_{2n}^2 + 16(4x-1)x^{n+1}W_{2n-2}^2 + (4x-1)W_0^2 - (4x-1)x(3W_0 - W_1)^2 + 2(W_1^2-2W_0^2-W_1W_0)(2^{2n}x^n-1)x$

and
if $(4x-1)(16x-1)(x-1)=0,$ *i.e.,* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k}^2 = \frac{\Psi}{3(64x^2-56x+7)}$$

where
$\Psi = (n(4x-1)(16x-17)+192x^2-168x+17)x^nW_{2n}^2+16(4(n+2)x-(n+1))x^nW_{2n-2}^2+4W_0^2-(8x-1)(3W_0-W_1)^2+2(W_1^2-2W_0^2-W_1W_0)(2^{2n}(n+1)x^n-1).$

(c) $(m=2,\ j=1)$
If $(4x-1)(16x-1)(x-1)\neq 0,$ *i.e.,* $x\neq\frac{1}{16},\ x\neq\frac{1}{4},\ x\neq 1,$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Delta}{(4x-1)(16x-1)(x-1)}$$

where
$\Delta = (4x-1)(16x-17)x^{n+1}W_{2n+1}^2+16(4x-1)x^{n+1}W_{2n-1}^2+(4x-1)W_1^2-4(4x-1)x(W_0-W_1)^2-4(W_1^2-2W_0^2-W_1W_0)(2^{2n}x^n-1)x$
and
if $(4x-1)(16x-1)(x-1)=0,$ *i.e.,* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k W_{2k+1}^2 = \frac{\Psi}{3(64x^2-56x+7)}$$

where
$\Psi = (n(4x-1)(16x-17)+192x^2-168x+17)x^nW_{2n+1}^2+16(4(n+2)x-(n+1))x^nW_{2n-1}^2+4W_1^2-4(8x-1)(W_0-W_1)^2-4(W_1^2-2W_0^2-W_1W_0)(2^{2n}(n+1)x^n-1).$

(d) $(m=-1,\ j=0)$
If $(x+2)(x-1)(x-4)\neq 0,$ *i.e.,* $x\neq -2,\ x\neq 1,\ x\neq 4,$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Delta}{(x+2)(x-1)(x-4)}$$

where
$\Delta = (x+2)x^{n+1}W_{-n+1}^2+(x+2)(x-5)x^{n+1}W_{-n}^2+4(x+2)W_0^2-(x+2)xW_1^2-4(W_1^2-2W_0^2-W_1W_0)((-2)^{-n}x^n-1)x$
and
if $(x+2)(x-1)(x-4)=0,$ *i.e.,* $x=-2$ *or* $x=1$ *or* $x=4$ *then*

$$\sum_{k=0}^{n} x^k W_{-k}^2 = \frac{\Psi}{3(x^2-2x-2)}$$

where
$\Psi = (n(x+2)+2(x+1))x^nW_{-n+1}^2+\big(n(x+2)(x-5)+3x^2-6x-10\big)x^nW_{-n}^2+4W_0^2-2(x+1)W_1^2-4(W_1^2-2W_0^2-W_1W_0)(x^n(-2)^{-n}(n+1)-1).$

(e) $(m=-2,\ j=0)$
If $(x-4)(x-1)(x-16)\neq 0,$ *i.e.,* $x\neq 1,\ x\neq 4,\ x\neq 16,$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Delta}{(x-4)(x-1)(x-16)}$$

where
$\Delta = (x-4)x^{n+1}W_{-2n+2}^2 + (x-4)(x-17)x^{n+1}W_{-2n}^2 + 16(x-4)W_0^2 - (x-4)\,xW_2^2 + 8(W_1^2 - 2W_0^2 - W_1W_0)(2^{-2n}x^n - 1)x$
and
if $(x-4)(x-1)(x-16) = 0$, *i.e.,* $x = 1$ *or* $x = 4$ *or* $x = 16$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k}^2 = \frac{\Psi}{3(x^2 - 14x + 28)}$$

where
$\Psi = (n(x-4)+2(x-2))x^nW_{-2n+2}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^nW_{-2n}^2 + 16W_0^2 - 2(x-2)W_2^2 + 8(W_1^2 - 2W_0^2 - W_1W_0)(2^{-2n}(n+1)x^n - 1).$

(f) $(m = -2,\ j = 1)$
If $(x-4)(x-1)(x-16) \neq 0$, *i.e.,* $x \neq 1,\ x \neq 4,\ x \neq 16$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Delta}{(x-4)(x-1)(x-16)}$$

where
$\Delta = (x-4)x^{n+1}W_{-2n+3}^2 + (x-4)(x-17)x^{n+1}W_{-2n+1}^2 + 16(x-4)W_1^2 - (x-4)\,x(2W_0 + 3W_1)^2 - 16(W_1^2 - 2W_0^2 - W_1W_0)(2^{-2n}x^n - 1)x$
and
if $(x-4)(x-1)(x-16) = 0$, *i.e.,* $x = 1$ *or* $x = 4$ *or* $x = 16$ *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1}^2 = \frac{\Psi}{3(x^2 - 14x + 28)}$$

where
$\Psi = (n(x-4)+2(x-2))x^nW_{-2n+3}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^nW_{-2n+1}^2 + 16W_1^2 - 2(x-2)(2W_0+3W_1)^2 - 16(W_1^2 - 2W_0^2 - W_1W_0)(2^{-2n}(n+1)x^n - 1).$

From the above proposition, we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n = J_n$ with $J_0 = 0, J_1 = 1$).

Corollary 2.7. *For* $n \geq 0$, *Jacobsthal numbers have the following properties:*

(a) $(m = 1,\ j = 0)$
If $(2x+1)(4x-1)(x-1) \neq 0$, *i.e.,* $x \neq -\frac{1}{2},\ x \neq \frac{1}{4},\ x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k J_k^2 = \frac{(2x+1)(4x-5)x^{n+1}J_n^2 + 4(2x+1)x^{n+1}J_{n-1}^2 - (2(-2)^n x^n + 2x - 1)x}{(2x+1)(4x-1)(x-1)}$$

and
if $(2x+1)(4x-1)(x-1) = 0$, *i.e.,* $x = -\frac{1}{2}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k J_k^2 = \frac{\Psi}{3(8x^2 - 4x - 1)}$$

where $\Psi = (n(2x+1)(4x-5) + 24x^2 - 12x - 5)x^nJ_n^2 + 4(2(n+2)x + (n+1))x^nJ_{n-1}^2 - (2(-2)^n(n+1)x^n + 4x - 1).$

(b) $(m = 2,\ j = 0)$

If $(4x-1)(16x-1)(x-1) \neq 0,$ *i.e.,* $x \neq \frac{1}{16},\ x \neq \frac{1}{4},\ x \neq 1,$ *then*

$$\sum_{k=0}^{n} x^k J_{2k}^2 = \frac{(4x-1)(16x-17)x^{n+1}J_{2n}^2 + 16(4x-1)x^{n+1}J_{2n-2}^2 - x\left(-2^{2n+1}x^n + 4x + 1\right)}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1) = 0,$ *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k J_{2k}^2 = \frac{\Psi}{3(64x^2 - 56x + 7)}$$

where $\Psi = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^n J_{2n}^2 + 16(4(n+2)x - (n+1))x^n J_{2n-2}^2 + 2^{2n+1}(n+1)x^n - 8x - 1.$

(c) $(m = 2,\ j = 1)$

If $(4x-1)(16x-1)(x-1) \neq 0,$ *i.e.,* $x \neq \frac{1}{16},\ x \neq \frac{1}{4},\ x \neq 1,$ *then*

$$\sum_{k=0}^{n} x^k J_{2k+1}^2 = \frac{(4x-1)(16x-17)x^{n+1}J_{2n+1}^2 + 16(4x-1)x^{n+1}J_{2n-1}^2 - 2^{2n+2}x^{n+1} - 16x^2 + 12x - 1}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1) = 0,$ *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k J_{2k+1}^2 = \frac{\Psi}{3(64x^2 - 56x + 7)}$$

where $\Psi = (n(4x-1)(16x-17) + 192x^2 - 168x + 17)x^n J_{2n+1}^2 + 16(4(n+2)x - (n+1))x^n J_{2n-1}^2 - 4(2^{2n}x^n(n+1) + 8x - 3)$

(d) $(m = -1,\ j = 0)$

If $(x+2)(x-1)(x-4) \neq 0,$ *i.e.,* $x \neq -2,\ x \neq 1,\ x \neq 4,$ *then*

$$\sum_{k=0}^{n} x^k J_{-k}^2 = \frac{(x+2)x^{n+1}J_{-n+1}^2 + (x+2)(x-5)x^{n+1}J_{-n}^2 - x(4(-2)^{-n}x^n + x - 2)}{(x+2)(x-1)(x-4)}$$

and

if $(x+2)(x-1)(x-4) = 0,$ *i.e.,* $x = -2$ *or* $x = 1$ *or* $x = 4$ *then*

$$\sum_{k=0}^{n} x^k J_{-k}^2 = \frac{\Psi}{3(x^2 - 2x - 2)}$$

where $\Psi = (n(x+2) + 2(x+1))x^n J_{-n+1}^2 + \left(n(x+2)(x-5) + 3x^2 - 6x - 10\right)x^n J_{-n}^2 - 4(n+1)(-2)^{-n}x^n + 2 - 2x.$

(e) $(m = -2,\ j = 0)$

If $(x-4)(x-1)(x-16) \neq 0,$ *i.e.,* $x \neq 1,\ x \neq 4,\ x \neq 16,$ *then*

$$\sum_{k=0}^{n} x^k J_{-2k}^2 = \frac{(x-4)x^{n+1}J_{-2n+2}^2 + (x-4)(x-17)x^{n+1}J_{-2n}^2 + 8x(x^n 2^{-2n} - 1) - x(x-4)}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k J_{-2k}^2 = \frac{\Psi}{3(x^2-14x+28)}$$

where $\Psi = (n(x-4)+2(x-2))x^n J_{-2n+2}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^n J_{-2n}^2 + 2(4x^n(n+1)2^{-2n}-x-2)$.

(f) $(m=-2,\ j=1)$

If $(x-4)(x-1)(x-16)\neq 0$, *i.e.,* $x\neq 1$, $x\neq 4$, $x\neq 16$, *then*

$$\sum_{k=0}^{n} x^k J_{-2k+1}^2 = \frac{(x-4)x^{n+1}J_{-2n+3}^2 + (x-4)(x-17)x^{n+1}J_{-2n+1}^2 - (2^{-2n+4}x^{n+1}+9x^2-68x+64)}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k J_{-2k+1}^2 = \frac{\Psi}{3(x^2-14x+28)}$$

where $\Psi = (n(x-4)+2(x-2))x^n J_{-2n+3}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^n J_{-2n+1}^2 - (n+1)2^{-2n+4}x^n - 18x + 68$.

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 2.8. *For* $n \geq 0$, *Jacobsthal-Lucas numbers have the following properties:*

(a) $(m=1,\ j=0)$

If $(2x+1)(4x-1)(x-1)\neq 0$, *i.e.,* $x\neq -\frac{1}{2}$, $x\neq \frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k j_k^2 = \frac{(2x+1)(4x-5)x^{n+1}j_n^2 + 4(2x+1)x^{n+1}j_{n-1}^2 + 18(-2)^n x^{n+1} - 2x^2 - 11x + 4}{(2x+1)(4x-1)(x-1)}$$

and

if $(2x+1)(4x-1)(x-1)=0$, *i.e.,* $x=-\frac{1}{2}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k j_k^2 = \frac{\Psi}{3(8x^2-4x-1)}$$

where $\Psi = (n(2x+1)(4x-5)+24x^2-12x-5)x^n j_n^2 + 4(2(n+2)x+(n+1))x^n j_{n-1}^2 + 18(n+1)(-2)^n x^n - 4x - 11$.

(b) $(m=2,\ j=0)$

If $(4x-1)(16x-1)(x-1)\neq 0$, *i.e.,* $x\neq \frac{1}{16}$, $x\neq \frac{1}{4}$, $x\neq 1$, *then*

$$\sum_{k=0}^{n} x^k j_{2k}^2 = \frac{(4x-1)(16x-17)x^{n+1}j_{2n}^2 + 16(4x-1)x^{n+1}j_{2n-2}^2 - 9\times 2^{2n+1}x^{n+1} - 100x^2 + 59x - 4}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1)=0$, *i.e.,* $x=\frac{1}{16}$ *or* $x=\frac{1}{4}$ *or* $x=1$ *then*

$$\sum_{k=0}^{n} x^k j_{2k}^2 = \frac{\Psi}{3(64x^2-56x+7)}$$

where $\Psi = (n(4x-1)(16x-17)+192x^2-168x+17)x^n j_{2n}^2 + 16(4(n+2)x-(n+1))x^n j_{2n-2}^2 - 9\times 2^{2n+1}(n+1)x^n - 200x + 59$.

(c) $(m = 2,\ j = 1)$

If $(4x-1)(16x-1)(x-1) \neq 0$, *i.e.,* $x \neq \frac{1}{16}$, $x \neq \frac{1}{4}$, $x \neq 1$, *then*

$$\sum_{k=0}^{n} x^k j_{2k+1}^2 = \frac{(4x-1)(16x-17)x^{n+1}j_{2n+1}^2 + 16(4x-1)x^{n+1}j_{2n-1}^2 + 9\times 2^{2n+2}x^{n+1} - 16x^2 - 28x - 1}{(4x-1)(16x-1)(x-1)}$$

and

if $(4x-1)(16x-1)(x-1) = 0$, *i.e.,* $x = \frac{1}{16}$ *or* $x = \frac{1}{4}$ *or* $x = 1$ *then*

$$\sum_{k=0}^{n} x^k j_{2k+1}^2 = \frac{\Psi}{3(64x^2 - 56x + 7)}$$

where $\Psi = (n(4x-1)(16x-17)+192x^2-168x+17)x^n j_{2n+1}^2 + 16(4(n+2)x-(n+1))x^n j_{2n-1}^2 + 9\times 2^{2n+2}(n+1)x^n - 32x - 28.$

(d) $(m = -1,\ j = 0)$

If $(x+2)(x-1)(x-4) \neq 0$, *i.e.,* $x \neq -2$, $x \neq 1$, $x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k j_{-k}^2 = \frac{(x+2)x^{n+1}j_{-n+1}^2 + (x+2)(x-5)x^{n+1}j_{-n}^2 + 36(-2)^{-n}x^{n+1} - x^2 - 22x + 32}{(x+2)(x-1)(x-4)}$$

and

if $(x+2)(x-1)(x-4) = 0$, *i.e.,* $x = -2$ *or* $x = 1$ *or* $x = 4$ *then*

$$\sum_{k=0}^{n} x^k j_{-k}^2 = \frac{\Psi}{3(x^2 - 2x - 2)}$$

where $\Psi = (n(x+2) + 2(x+1))x^n j_{-n+1}^2 + \left(n(x+2)(x-5) + 3x^2 - 6x - 10\right)x^n j_{-n}^2 + 36(n+1)(-2)^{-n}x^n - 2x - 22.$

(e) $(m = -2,\ j = 0)$

If $(x-4)(x-1)(x-16) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 4$, $x \neq 16$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k}^2 = \frac{(x-4)x^{n+1}j_{-2n+2}^2 + (x-4)(x-17)x^{n+1}j_{-2n}^2 - 72\times 2^{-2n}x^{n+1} - 25x^2 + 236x - 256}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16) = 0$, *i.e.,* $x = 1$ *or* $x = 4$ *or* $x = 16$ *then*

$$\sum_{k=0}^{n} x^k j_{-2k}^2 = \frac{\Psi}{3(x^2 - 14x + 28)}$$

where $\Psi = (n(x-4) + 2(x-2))x^n j_{-2n+2}^2 + (n(x-4)(x-17) + 3x^2 - 42x + 68)x^n j_{-2n}^2 - 72(n+1)2^{-2n}x^n - 50x + 236.$

(f) $(m = -2,\ j = 1)$

If $(x-4)(x-1)(x-16) \neq 0$, *i.e.,* $x \neq 1$, $x \neq 4$, $x \neq 16$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1}^2 = \frac{(x-4)x^{n+1}j_{-2n+3}^2 + (x-4)(x-17)x^{n+1}j_{-2n+1}^2 + 144x^{n+1}2^{-2n} - 49x^2 + 68x - 64}{(x-4)(x-1)(x-16)}$$

and

if $(x-4)(x-1)(x-16)=0$, *i.e.,* $x=1$ *or* $x=4$ *or* $x=16$ *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1}^2 = \frac{\Psi}{3(x^2-14x+28)}$$

where $\Psi = (n(x-4)+2(x-2))x^n j_{-2n+3}^2 + (n(x-4)(x-17)+3x^2-42x+68)x^n j_{-2n+1}^2 + 144(n+1)2^{-2n}x^n - 98x + 68$.

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 2.9. *For* $n \geq 0$, *Jacobsthal numbers and Jacobsthal-Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} J_k^2 = \frac{1}{9}(-(3n-7)J_n^2 + 4(3n+5)J_{n-1}^2 - 2(n+1)(-2)^n - 3)$.
- **(b)** $\sum_{k=0}^{n} J_{2k}^2 = \frac{1}{45}(-(3n-41)J_{2n}^2 + 16(3n+7)J_{2n-2}^2 + (n+1)2^{2n+1} - 9)$.
- **(c)** $\sum_{k=0}^{n} J_{2k+1}^2 = \frac{1}{45}(-(3n-41)J_{2n+1}^2 + 16(3n+7)J_{2n-1}^2 - (n+1)2^{2n+2} - 20)$.
- **(d)** $\sum_{k=0}^{n} J_{-k}^2 = \frac{1}{9}(-(3n+4)J_{-n+1}^2 + (12n+13)J_{-n}^2 + 4(n+1)(-2)^{-n})$.
- **(e)** $\sum_{k=0}^{n} J_{-2k}^2 = \frac{1}{45}(-(3n+2)J_{-2n+2}^2 + (48n+29)J_{-2n}^2 + (n+1)2^{-2n+3} - 6)$.
- **(f)** $\sum_{k=0}^{n} J_{-2k+1}^2 = \frac{1}{45}(-(3n+2)J_{-2n+3}^2 + (48n+29)J_{-2n+1}^2 - (n+1)2^{-2n+4} + 50)$.

2.

- **(a)** $\sum_{k=0}^{n} j_k^2 = \frac{1}{9}(-(3n-7)j_n^2 + 4(3n+5)j_{n-1}^2 + 18(n+1)(-2)^n - 15)$.
- **(b)** $\sum_{k=0}^{n} j_{2k}^2 = \frac{1}{45}(-(3n-41)j_{2n}^2 + 16(3n+7)j_{2n-2}^2 - 9(n+1)2^{2n+1} - 141)$.
- **(c)** $\sum_{k=0}^{n} j_{2k+1}^2 = \frac{1}{45}(16(3n+7)j_{2n-1}^2 - (3n-41)j_{2n+1}^2 + 9(n+1)2^{2n+2} - 60)$.
- **(d)** $\sum_{k=0}^{n} j_{-k}^2 = \frac{1}{9}(-(3n+4)j_{-n+1}^2 + (12n+13)j_{-n}^2 - 36(n+1)(-2)^{-n} + 24)$.
- **(e)** $\sum_{k=0}^{n} j_{-2k}^2 = \frac{1}{45}(-(3n+2)j_{-2n+2}^2 + (48n+29)j_{-2n}^2 - 9(n+1)2^{-2n+3} + 186)$.
- **(f)** $\sum_{k=0}^{n} j_{-2k+1}^2 = \frac{1}{45}(-(3n+2)j_{-2n+3}^2 + (48n+29)j_{-2n+1}^2 + 9(n+1)2^{-2n+4} - 30)$.

3 Conclusion

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Fibonacci-Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences. All the listed identities in the propositions and corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered.

We can mention some applications of sum formulas. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences.

Competing Interests

Author has declared that no competing interests exist.

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