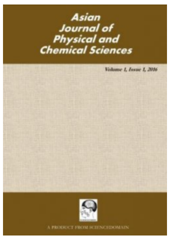

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Equivalent Multiple Complex SUSY For Real SUSY

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Author's contribution

The sole author designed, analyzed and interpreted and prepared the manuscript.

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ABSTRACT

We notice real $SUSY$ Hamiltonians have multiple equivalent complex Hamiltonians which may be (i) $\mathcal{PT}$ invariant (ii) $\mathcal{T}$ invariant or (iii) combination of both in nature . These three types of complex Hamiltonians give the same energy spectrum . We present here analytical results for the exactly solvable system and numerical results for others.

Keywords: Supersymmetry; $\mathcal{PT}$ symmetry; real spectra; complex hamiltonians.

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1 INTRODUCTION

Our understanding on real spectra in quantum physics has been drastically changed after the thought breaking idea of Bender and Boettecher [1], who introduced the concept of $\mathcal{PT}$ symmetry . The operator $\mathcal{P}$ stands for parity ,reflecting the behaviour :$x \rightarrow -x$;$p \rightarrow -p$ and $i \rightarrow i$. Similarly the operator $\mathcal{T}$ represents time reversal ,reflecting the behaviour :$x \rightarrow x$;$p \rightarrow -p$ and $i \rightarrow -i$. However , $\mathcal{PT}$ symmetry understanding took a new turn when Jones and Mateo [2],theoretically proved that inverted quartic oscillator

$$H = p^2 - x^4 \tag{1}$$

has equivalent hermitian operator reflecting the iso-spectral character . Later on Nanayakkara and Mathanaranjan [3]noticed that one dimensional complex Hamiltonian

$$H = p^2 - x^4 + 4ix \tag{2}$$

also posseses equivalent hermitian operator reflecting iso-spectra . One simple question comes to mind that whether real Hamiltonians have complex counter part reflecting iso-spectral behaviour ? In order to address this question we consider supersymmetry as an ideal example .

2 REAL SUSY

Here, we simply consider, SUSY Hamiltonians in short as follows . The generated Hamiltonians in terms of superpotential W, can be written as [4-7]

$$H^+ = p^2 + \frac{dW(x)}{dx} + W^2 \tag{3}$$

and

$$H^- = p^2 - \frac{dW(x)}{dx} + W^2 \tag{4}$$

For SUSY energy conditions

$$E_n^{(+)} = E_{n+1}^{(-)} \tag{5}$$

with

$$E_0^{(-)} = 0 \tag{6}$$

let us consider two quadratic exactly solvable Hamiltonians as

$$H^- = p^2 + x^2 - 1 \tag{7}$$

$$H^+ = p^2 + x^2 + 1 \tag{8}$$

We can have another two Hamiltonians as

$$H^- = p^2 + x^6 + 2x^4 - 2x^2 - 1 \tag{9}$$

$$H^+ = p^2 + x^6 + 2x^4 + 4x^2 + 1 \tag{10}$$

The computed eigenvalues are tabulated in table 1 using matrix diagonalisation method [8]. Further ,for the iso-spectral energy condition

$$E_n^{(+)} = E_n^{(-)} \tag{11}$$

we can have the Hamiltonians as,

$$H^- = p^2 + x^4 + x^2 - 2x + 0.25 \tag{12}$$

$$H^+ = p^2 + x^4 + x^2 + 2x + 0.25 \tag{13}$$

3 COMPLEX SUSY

Before, going to introduce complex SUSY , we would like to bring the attention of reader, an interesting idea on complex transformation of momentum [9-12] in terms of co-ordinate as

$$p \rightarrow p + ix \tag{14}$$

It has been explicitly addressed in the case of Harmonic Oscillator[9-12] . In this paper we apply the same to real SUSY operators as

$$H_1^+ = p^2 - x^2 + i(xp + px) + \frac{dW(x)}{dx} + W^2 \tag{15}$$

and

$$H_1^- = p^2 - x^2 + i(xp + px) - \frac{dW(x)}{dx} + W^2 \tag{16}$$

The above two Hamiltonians are $\mathcal{PT}$ invariant in nature . In our view ,the above two Hamiltonians must retain iso-spectral behaviour with that of real SUSY Hamiltonians .Now consider another transformation as

$$p \rightarrow p + i \tag{17}$$

The new complex Hamiltonians are as follows

$$H_2^+ = p^2 + 2ip - 1 + \frac{dW(x)}{dx} + W^2 \tag{18}$$

and

$$H_2^- = p^2 + 2ip - 1 - \frac{dW(x)}{dx} + W^2 \tag{19}$$

The above two Hamiltonians are $\mathcal{T}$ invariant in nature . Now consider combination of these two and write two new Hamiltonians as

$$H_3^+ = p^2 - x^2 + i(xp + px) + 2ip - 2x - 1 + \frac{dW(x)}{dx} + W^2 \tag{20}$$

and

$$H_3^- = p^2 - x^2 + i(xp + px) + 2ip - 2x - 1 - \frac{dW(x)}{dx} + W^2 \tag{21}$$

Interestingly in this case the Hamiltonians are neither $\mathcal{PT}$ invariant nor$\mathcal{T}$ invariant in nature . In order to show explicitly we consider few cases as given below.

3.1 Complex SUSY: Analytical Result

Here we would like to state that quadratic operator can be addressed analytically.Let us discuss few lines on analytical expression for energy level relating to quadratic Hamiltonian [10,11]

$$H = h_{11}p^2 + ih_{12}(xp + px) + h_{22}x^2 + ih_1 p + h_2 x \tag{22}$$

having eigenvalue

$$\epsilon_n = [\sqrt{(h_{11}h_{22} + h_{12}^2)}](2n + 1) + \frac{(h_1^2 h_{22} - h_2^2 h_{11} - 2h_1 h_2 h_{12})}{4(h_{11}h_{22} + h_{12}^2)} \tag{23}$$

Here we suggest two different complex Hamiltonians as follows Now consider complex SUSY on exactly solvable real systems as:

$$H_1^+ = p^2 + i(xp + px) + 1 \tag{24}$$

$$H_1^- = p^2 + i(xp + px) - 1 \tag{25}$$

$$H_2^+ = p^2 + x^2 + 2ip \tag{26}$$

$$H_2^- = p^2 + x^2 + 2ip - 2 \tag{27}$$

$$H_3^+ = p^2 + i(xp + px) + 2ip - 2x \tag{28}$$

$$H_3^- = p^2 + i(xp + px) + 2ip - 2x - 2 \tag{29}$$

Using the above exression ,one can see that

$$H_3^-, H_2^-, H_1^- = 2n \tag{30}$$

and

$$H_3^+, H_2^+, H_1^+ = 2n + 2 \tag{31}$$

Here , n=0,1,2,3 ... . Interested reader can easily verify the SUSY energy conditions.

3.2 Complex SUSY: Numerical Result

Here we consider the complex SUSY Hamiltonians as

$$H_1^- = p^2 + x^6 + 2x^4 - 3x^2 - 1 + i(xp + px) \tag{32}$$

$$H_1^+ = p^2 + x^6 + 2x^4 + 3x^2 + 1 + i(xp + px) \tag{33}$$

$$H_2^- = p^2 + x^6 + 2x^4 - 2x^2 + 2ip - 2 \tag{34}$$

$$H_2^+ = p^2 + x^6 + 2x^4 + 4x^2 + 2ip \tag{35}$$

$$H_3^- = p^2 + x^6 + 2x^4 - 3x^2 - 2 + i(xp + px) + 2ip - 2x \tag{36}$$

$$H_3^+ = p^2 + x^6 + 2x^4 + 3x^2 + i(xp + px) + 2ip - 2x \tag{37}$$

The above Hamiltonians can not be solved analytically. For numerical results we apply matrix diagonalisation method [8] as follows

$$H|\Psi>= E|\Psi> \tag{38}$$

where

$$|\Psi>=\sum_m A_m|m> \tag{39}$$

In the above $|m>$ is the harmonic oscillator wave function which satisfies the eigenvalue relation

$$(p^2+x^2)|m>=(2m+1)|m> \tag{40}$$

Further in general ,for SUSY Hamiltonian we get nine term recurrence relation as

$$A_{m-6}P_m+A_{m-4}Q_m+A_{m-2}R_m+A_{m-1}S_m+A_mT_m+A_{m+1}U_m+A_{m+2}V_m+A_{m+4}W_m+A_{m+6}Y_m=0 \tag{41}$$

Where

$$P_m=<m|H|m-6> \tag{42}$$
$$Q_m=<m|H|m-4> \tag{43}$$
$$R_m=<m|H|m-2> \tag{44}$$
$$S_m=<m|H|m-1> \tag{45}$$
$$U_m=<m|H|m+1> \tag{46}$$
$$V_m=<m|H|m+2> \tag{47}$$
$$W_m=<m|H|m+4> \tag{48}$$
$$Y_m=<m|H|m+6> \tag{49}$$
$$T_m=<m|H|m>-E \tag{50}$$

For the benefit of readers we present diagonal elements as given below .

$$<m|H^+|m>=2.5m^3+6.75m^2+13m+6.875 \tag{51}$$
$$<m|H_1^+|m>=2.5m^3+6.75m^2+12m+6.375 \tag{52}$$
$$<m|H_2^+|m>=2.5m^3+6.75m^2+13m+5.875 \tag{53}$$
$$<m|H_3^+|m>=2.5m^3+6.75m^2+12m+5.375 \tag{54}$$
$$<m|H_1^-|m>=2.5m^3+6.75m^2+6m+1.375 \tag{55}$$
$$<m|H_2^-|m>=2.5m^3+6.75m^2+7m+0.875 \tag{56}$$
$$<m|H_3^-|m>=2.5m^3+6.75m^2+6m+0.375 \tag{57}$$
$$<m|H^-|m>=2.5m^3+6.75m^2+7m+1.875 \tag{58}$$

In table 1 , we reflect eigenvalues along with the real SUSY Hamiltonians . Here $H^-_{1,2,3}\rightarrow E_n^{-C}$ and $H^+_{1,2,3}\rightarrow E_n^{+C}$

Table 1. Eigenvalues of real and complex SUSY hamiltonians

n	$E_n^{(-R)}$	$E_n^{(+R)}$	E_n^{-C}	E_n^{+C}
0	0	3.373 001 0	0	3.373 001 0
1	3.373 001 0	8.743 633 3	3.373 001 0	8.743 633 3
2	8.743 633 3	15.261 907 1	8.743 633 3	15.261 907 1
3	15.261 907 1	22.749 693 9	15.261 907 1	22.749 693 9

3.3 Iso- Spectral Complex Hamiltonians: Numerical Result

Now we consider iso-spectral nature of complex SUSY Hamiltonians . The Hamiltonians considered here as

$$H_1^- = p^2 + x^4 + i(xp + px) - 2x + 0.25 \tag{59}$$

$$H_1^+ = p^2 + x^4 + i(xp + px) + 2x + 0.25 \tag{60}$$

$$H_2^- = p^2 + x^4 + x^2 + 2ip - 2x - 0.75 \tag{61}$$

$$H_2^+ = p^2 + x^4 + x^2 + 2ip + 2x - 0.75 \tag{62}$$

$$H_3^- = p^2 + x^4 + i(xp + px) - 4x + 2ip - 0.75 \tag{63}$$

$$H_3^+ = p^2 + x^4 + i(xp + px) + 2ip - 0.75 \tag{64}$$

Here we calculate energy eigenvalues using matrix diagonalisation , on solving the eigenvalue relation as stated above . Here we solve a seven term recurrence relation as given below

$$A_{m-4}Q_m + A_{m-2}R_m + A_{m-1}S_m + A_mT_m + A_{m+1}U_m + A_{m+2}V_m + A_{m+4}W_m = 0 \tag{65}$$

where

$$Q_m =< m|H|m-4 > \tag{66}$$

$$R_m =< m|H|m-2 > \tag{67}$$

$$S_m =< m|H|m-1 > \tag{68}$$

$$U_m =< m|H|m+1 > \tag{69}$$

$$V_m =< m|H|m+2 > \tag{70}$$

$$W_m =< m|H|m+4 > \tag{71}$$

$$T_m =< m|H|m > -E \tag{72}$$

For the benefit of readers we present diagonal elements as given below .

$$< m|H^+|m >= 1.5m^2 + 3.5m + 2 \tag{73}$$

$$< m|H_1^+|m >= 1.5m^2 + 2.5m + 1.5 \tag{74}$$

$$< m|H_2^+|m >= 1.5m^2 + 3.5m + 1 \tag{75}$$

$$< m|H_3^+|m >= 1.5m^2 + 2.5m + 0.5 \tag{76}$$

$$< m|H_1^-|m >= 1.5m^2 + 2.5m + 1.5 \tag{77}$$

$$< m|H_2^-|m >= 1.5m^2 + 3.5m + 1 \tag{78}$$

$$< m|H_3^-|m >= 1.5m^2 + 2.5m + 0.5 \tag{79}$$

$$< m|H^-|m >= 1.5m^2 + 3.5m + 2 \tag{80}$$

In table 2, we reflect eigenvalues along with the real iso-spectral Hamiltonians using matrix diagonalisation method as described earlier . Here $H_{1,2,3}^- \to E_n^{-C}$ and $H_{1,2,3}^+ \to E_n^{+C}$

Table 2. Eigenvalues of real and complex Iso-Spectral hamiltonians

n	$E_n^{(-R)}$	$E_n^{(+R)}$	E_n^{-C}	E_n^{+C}
0	1.277 243 8	1.277 243 8	1.277 243 8	1.277 243 8
1	4.771 390 0	4.771 390 0	4.771 390 0	4.771 390 0
2	8.812 448 7	8.812 448 7	8.812 448 7	8.812 448 7
3	13.333 679 9	13.333 679 9	13.333 679 9	13.333 679 9

4 CONCLUSION

In this paper we have found that for real $SUSY$ Hamiltonians, there are multiple complex equivalent Hamiltonians reflecting iso-spectral behaviour. Hence we believe all bounded operators are associated with equivalent complex operators i.e

$$\text{Bounded Operator}$$
$$\Updownarrow$$
$$\overbrace{\mathcal{PT}+\mathcal{T}+\mathcal{PT}\sim\mathcal{T}}$$

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COMPETING INTERESTS

Author has declared that no competing interests exist.

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