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Computation of Eigenvalues of the Fourth Order Sturm-Liouville BVP by Galerkin Weighted Residual Method

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Abstract

The aim of this paper is to compute eigenvalues of fourth order regular Sturm-Liouville Boundary Value Problems (SLP). We propose the Galerkin weighted residual method with Bernstein polynomials as basis functions to approximate the solutions of SLP. We derive rigorous matrix formulations to compute the eigenvalues of the SLP. Special care has been given about how the polynomials satisfy the corresponding homogeneous form of *Dirichlet* boundary conditions. The approximate eigenvalues are compared with the exact result and also compared with the relevant studies by some authors. The results in this study agree with that of the other relevant articles.

Keywords: Galerkin method; Bernstein polynomials; Sturm-Liouville problems; Eigenvalue.

1 Introduction

The concept of eigenvalue problem is rather important both in pure and applied mathematics, a physical system, such as pendulum, a vibrating or rotating shaft etc. The physical system such as pendulums and vibrating and rotating shafts are connected with eigenpairs of the system. The Sturm-Liouville systems arise from vibration problems in continuum mechanics.

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In the literature, we observe that many researchers studied second order Sturm Liouville eigenvalue problems. Some authors Abbasbandy and Shirazdi [1], Shi and Cao [2], Yucel and Boubaker [3], Gamel and Sameeh [4], Taher et al. [5] paid their attention to develop various techniques for finding eigenvalues of fourth order Sturm-Liouville BVP's. They applied different algorithms to minimize the convergence rates.

Chanane and Chanane [6,7] introduced a novel series representation for the boundary/characteristic function associated with fourth-order Sturm-Liouville problems using the concepts of Fliess series and iterated integrals. Chawla [8] presented fourth-order finite-difference method for computing eigenvalues of fourth-order two-point boundary value problems. Usmani and Sakai [9] applied finite difference method of order two and four for computing eigenvalues of the fourth-order boundary value problems while Twizell and Matar [10] developed finite difference method for approximating the eigenvalues of fourth-order boundary value problems.

Jia et al. [11] approximated the eigenvalues of fourth order BVP for a class of crosswise vibration equation of beam using Galerkin method and obtained the estimation of errors using the trigonometric polynomials that satisfies all the boundary conditions directly. The Adomian decomposition method (ADM) to solve fourth-order eigenvalue problems was used by Attili and Lesnic [12]. Syam and Siyyam [13] developed a variational iteration technique (VIM) for finding the eigenvalues of fourth-order non-singular Sturm-Liouville problems. Recently, Chanane [14] has enlarged the scope of the Extended Sampling method [15] which was devised initially for second-order Sturm-Liouville (SLE) problems to fourth-order ones. Abbasbandy and Shirzadi [1] applied the homotopy analysis method (HAM) to numerically approximate the eigenvalues of the second and fourth order Sturm-Liouville problems. Shi and Cao [2] presented a computational method for solving eigenvalue problems of high-order ordinary differential equations which based on the use of Haar wavelets. Yucel and Boubaker [3] applied differential quadrature method (DQM) and boubaker polynomial expansion scheme (BPES) for efficient computation of the eigenvalues of fourth-order Sturm-Liouville problems. Gamel and Sameeh [4] applied Chebychev method for finding eigenvalues of fourth order nonsingular Sturm-Liouville problems and compared the results to the other methods available in the literature.

Very recently Taher et al. [5] applied an efficient technique using Chebychev spectral collocation method where Chebychev differentiation matrix is defined and computed the eigenvalues of SLP's. Since Bernstein polynomials have been used for the solution of differential equations by Doha et al. [16] and also by Islam and Hossain [17], this partially motivates our interest to compute the eigenvalues of the SLP's using Bernstein polynomials. Another motivation is concerned with Galerkin weighted residual method which can provide solutions to many complicated problems.
We organize this article as follows.

We give a brief introduction of Bernstein polynomials in section 2 along with their properties. The formulation of the general linear fourth order Sturm-Liouville problems by utilizing the technique of Galerkin weighted residual method incorporated with the boundary conditions have been discussed in Section 3. In Section 4, we consider numerical examples to verify the efficiency of the proposed method.

2 Bernstein Polynomials

The general form of the Bernstein polynomials of *n*- th degree over the interval $[a,b]$ defined by Islam and Hossain [17].

$$B_{i,n}(x) = \binom{n}{i}\frac{(x-a)^{i}(b-x)^{n-i}}{(b-a)^{n}} \quad , \; a \le x \le b, \quad i = 0,1,2\cdots\cdots,n. \tag{1}$$

Now the addition of two polynomials of degree *n*-1 over the interval [0, *L*] as [16]

$$B_{i,n}(x) = \frac{(L-x)}{L} B_{i,n-1}(x) + \frac{x}{L} B_{i-1,n-1}(x)$$

and the produact of two polynomials is defined as

$$B_{i,j}(x)\, B_{k,l}(x) = \frac{\binom{j}{i}\binom{l}{k}}{\binom{j+l}{i+k}} B_{i+k,\,j+l}(x)$$

Also the first derivative and second derivatives may be defined successively, as

$$B'_{i,n}(x) = \frac{n}{L}\left(B_{i-1,n-1}(x) - B_{i,n-1}(x)\right)$$

and

$$B''_{i,n}(x) = \frac{n(n-1)}{L^2}\left[B_{i-2,n-2}(x) - 2\,B_{i-1,n-2}(x) + B_{i,n-2}\right]$$

Note that each these n+1 polynomials satisfies the following properties

i) $B_{i,n}(x) = 0$, if $i < 0 \;\; or \;\; i > n$,

ii) $B_{i,n}(a) = 0 = B_{i,n}(b)$, $1 \le i \le n-1$

iii) $\sum_{i=0}^{n} B_{i,n}(x) = 1$

For simplicity we denote $B_{i,n}(x)$ as B_i throughout the paper. Since the piecewise polynomials are differentiable and integrable, Bernstein polynomials defined in equation (1) form a complete basis over the finite interval.

3 Matrix Formulation

Consider the following general fourth order nonsingular Sturm-Liouville problem (SLP)

$$\left(p(x)u''(x)\right)'' - \left(q(x)u'(x)\right)' + r(x)u = \lambda\mu(x)u\,, \quad 0 < x < L\,. \tag{2}$$

Here L is finite number; $p(x), q(x), r(x)$ and $\mu(x)$ are all piecewise continuous functions and $p(x)$, $\mu(x) > 0$ subject to some specified conditions and at these conditions mean that equation (2) is regular, i.e., nonsingular.

We can rewrite the equation (2) in the following form as a general fourth order Sturm-Liouville problems (SLP)

$$u^{(4)}(x) + a_3(x)u^{(3)} + a_2(x)u'' + a_1(x)u' + a_0(x)u = \lambda\phi(x)u, \tag{3}$$

where $a_3(x) = \frac{2p'(x)}{p(x)}$, $a_2(x) = \frac{p''(x) - q(x)}{p(x)}$, $a_1(x) = -\frac{q'(x)}{p(x)}$, $a_0(x) = -\frac{r(x)}{p(x)}$,
$\phi(x) = \frac{\mu(x)}{p(x)}$

Let us consider the fourth order SLP (3) subject to the boundary conditions

$$u(a) = 0, \quad u(b) = 0\ ; \quad u'(a) = 0 \quad u'(b) = 0 \tag{4}$$

To approximate the solution of SLP (3), we express in terms of Bernstein polynomial basis as

$$\tilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_i(x) \tag{5}$$

where $\theta_0(x)$ is specified by the Dirichlet boundary conditions and $B_i(a) = 0$ and $B_i(b) = 0$ for each $i = 1,2,3,\ldots\ldots,n-1$.

Using (5) into equation (3), the Galerkin weighted residual equations are [18]:

$$\int_a^b \left[\tilde{u}^{(4)}(x) + a_3(x)\tilde{u}^{(3)} + a_2(x)\tilde{u}'' + a_1(x)\tilde{u}' + a_0(x)\tilde{u} - \lambda\phi(x)\tilde{u}\right]B_j\, dx = 0\ , \quad j = 1,2,3,\cdots\cdots,n. \tag{6}$$

Now integrating each term of (6) by parts, we have

$$\begin{aligned}\int_a^b \tilde{u}^{(4)}(x)B_j(x)dx &= \left[B_j(x)\tilde{u}^{(3)}(x)\right]_a^b - \int_a^b B_j'(x)\tilde{u}^{(3)}(x)dx \\ &= -\left[B_j'(x)\tilde{u}''(x)\right]_a^b + \int_a^b B_j''(x)\tilde{u}''(x)dx \\ &= -\left[B_j'(x)\tilde{u}''(x)\right]_a^b - \int_a^b B_j^{(3)}\tilde{u}'dx \end{aligned} \tag{7}$$

Since $\left[B_j(x)\tilde{u}^{(3)}(x)\right]_a^b = 0$ by the Dirichlet boundary conditions.

Similarly,

$$\begin{aligned}\int_a^b a_3(x)\tilde{u}^{(3)}B_j(x)dx &= \left[a_3(x)B_j(x)\tilde{u}''(x)\right]_a^b - \int_a^b \left[a_3(x)B_j(x)\right]'\tilde{u}''dx \\ &= -\left[\left(a_3(x)B_j(x)\right)'\tilde{u}'(x)\right]_a^b + \int_a^b \left[a_3(x)B_j(x)\right]''\tilde{u}'\,dx \\ &= -\left[\left(a_3(x)B_j(x)\right)'\tilde{u}'(x)\right]_{x=b} + \left[\left(a_3(x)B_j(x)\right)'\tilde{u}'(x)\right]_{x=a} + \int_a^b \left[a_3(x)B_j(x)\right]''\tilde{u}'\,dx \\ &= \int_a^b \left[a_3(x)B_j(x)\right]''\tilde{u}'\,dx \end{aligned} \tag{8}$$

Equations (7) and (8) are obtained by imposing boundary conditions in equation (4).

Also,

$$\int_a^b a_2(x)\tilde{u}''B_j(x)dx = \left[a_2(x)B_j(x)\tilde{u}'(x)\right]_a^b - \int_a^b \left[a_2(x)B_j(x)\right]'\tilde{u}'dx$$

$$= -\int_a^b \left[a_2(x)B_j(x)\right]'\tilde{u}'dx \tag{9}$$

$$\int_a^b a_1(x)\tilde{u}'(x)B_j(x)dx = \left[a_1(x)B_j(x)\tilde{u}(x)\right]_a^b - \int_a^b \left[a_1(x)B_j(x)\right]'\tilde{u}(x)dx.$$

$$= -\int_a^b \left[a_1(x)B_j(x)\right]'\tilde{u}dx \tag{10}$$

Inserting $B_j(a) = B_j(b) = 0$ in the above integrals, we finally obtain the equations (7), (8),(9) and (10)

Substituting (7), (8), (9) and (10) into (6) and after rearranging the terms we have

$$\int_a^b \Big[-B_j^{(3)}(x)\tilde{u}' + \left[a_3(x)B_j(x)\right]''\tilde{u}' - \left[a_2(x)B_j(x)\right]'\tilde{u}'$$
$$-\left[a_1(x)B_j(x)\right]'\tilde{u} + a_0(x)B_j(x)\tilde{u} - \lambda\phi(x)B_j\tilde{u}\Big]dx - \left[B_j'(x)\tilde{u}''(x)\right]_a^b = 0 \tag{11}$$

Also from equation (5), we have

$$\tilde{u}(a) = \sum_{i=1}^{n-1} c_i B_i(a), \quad \tilde{u}(b) = \sum_{i=1}^{n-1} c_i B_i(b) \tag{12a}$$

$$\tilde{u}''(x) = \sum_{i=1}^{n-1} c_i B_i''(x) \tag{12b}$$

Using equations (12a) and (12b) into equation (11) we obtain

$$\sum_{i=1}^{n-1} \Bigg[\int_a^b \Big[-B_j^{(3)}(x)B_i'(x) + \left[a_3(x)B_j(x)\right]''B_i'(x) - \left[a_2(x)B_j(x)\right]'B_i'(x)$$
$$-\left[a_1(x)B_j(x)\right]'B_i(x) + a_0(x)B_i(x)B_j(x)\Big]dx\Bigg]c_i$$
$$+\sum_{i=1}^{n-1} \left\{ -\left[B_j'(x)B_i''(x)\right]_{x=b} + \left[B_j'(x)B_i''(x)\right]_{x=a} \right\} c_i$$
$$= \lambda \sum_{i=1}^{n-1} \int_a^b \left[\phi(x)B_iB_j dx\right] c_i \tag{13}$$

Finally, the eigenvalues are obtained in matrix form as below

$$\sum_{i=1}^{n-1}\left[D_{i,j} - \lambda F_{i,j}\ \right]c_i = 0 \tag{14a}$$

where,

$$D_{i,j} = \int_a^b \left\{ -B_j^{(3)}(x)\,B_i'(x) + \left[a_3(x)B_j(x)\right]''B_i'(x) - \left[a_2(x)B_j(x)\right]'B_i'(x)\right\}dx$$
$$+\int_a^b \left\{ -\left[a_1(x)B_j(x)\right]'B_i(x) + a_0(x)B_i(x)B_j(x)\right\}dx - \left[B_j'(x)B_i''(x)\right]_{x=b} \tag{14b}$$

$$F_{i,j} = \int_a^b \left[\phi(x)\,B_iB_j dx\right] \tag{14c}$$

Hence, the eigenvalues can be obtained by solving the determinant of the coefficient matrix in equation (14a) such that

$$\det\left(D_{i,j} - \lambda F_{i,j}\right) = 0. \tag{15}$$

Similarly for the boundary conditions of the type: $u(a) = 0,\quad u(b) = 0\ ,\quad u''(a) = 0\ ,\quad u''(b) = 0$, the formulation can be obtained easily.

4 Test Examples

In this section we present five numerical examples of fourth order SLP problems, using the method outlined in the previous section. All the numerical calculations are carried out using MATLAB 13 by an itel(R) Core(TM) i5-4570 CPU with power 3.20 GHz CPU, equipped with 8 GB of Ram. The convergence of the Galerkin method is measured by the relative error

$$\varepsilon_k = \left|\frac{\lambda^{Exact} - \lambda^{(Gal)}}{\lambda^{exact}}\right| < 10^{-10}. \tag{16}$$

Example 1(a): We first consider the Sturm-Liouville BVP examined by Yucel and Boubaker [3], Gamel and Sameeh [4] and Taher *et a*l [5] .

$$u^{(4)}(x) - \lambda u(x) = 0, \qquad 0 < x < 1 \tag{17a}$$

$$\left.\begin{array}{l} u(0) = u'(0) = 0 \\ u(1) = u''(1) = 0 \end{array}\right\}, \tag{17b}$$

which corresponds to the case $a_0(x) = a_1(x) = a_2(x) = a_3(x) = 0$, $a = 0$ and $b = 1$ in equation (3).

The exact solution of (17a) is obtained by solving

$$\tanh\left(\sqrt{\lambda}\right) - \tan\left(\sqrt{\lambda}\right) = 0 . \tag{18}$$

Using the method illustrated in section 3, we approximate $u(x)$ as

$$\widetilde{u}(x) = \theta_0(x) + \sum_{i=1}^{n-1} c_i B_i(x). \tag{19}$$

Here $\theta_0(x) = 0$ as specified by the Dirichlet boundary conditions of equation (17b).

The weighted residual, equation (17a) becomes

$$\sum_{i=1}^{n-1} \left[D_{i,j} - \lambda F_{i,j}\right] c_i = 0 \ , \ j = 1, 2, 3, \ldots\ldots\ldots, n-1 \tag{20a}$$

where,

$$D_{i,j} = \int_0^1 - B_j^{(3)} B_i' dx + \left[B_i''(0) B_j'(0) - B_j''(1) B_i'(1)\right] \tag{20b}$$

$$F_{i,j} = \int_0^1 B_i \, B_j dx \tag{20c}$$

Solving the determinant of the system in (20a), we get the approximate eigenvalues for different values of *n*.

Exact eigenvalues and relative errors are tabulated in Table 1 using different degrees of polynomials with the relative error for the differential quadrature method [3], Chebychev method [4] and Chebychev spectral collocation method [5].

The results, obtained using n=20, for the first six eigenvalues of the problem using Bernstein polynomials are shown in Table 2. The observed CPU time is 3.78 seconds.

Example 1(b): Consider the Sturm-Liouville BVP worked out by Gamel and Sameeh [4], Syam & Siyyam [13].

$$u^{(4)}(x) - \lambda u(x) = 0, \tag{21a}$$

$$u''(0) = u''(1) = u'''(0) = u'''(1) = 0. \tag{21b}$$

Table 3 shows the comparison of our result obtained using $n=22$, for Bernstein polynomial, with the first six eigenvalues of the problem with Gamel and Sameeh [4], Syam and Siyyam [13].

Table 1. Observed relative errors of the eigenvalues for example 1(a)

Exact eigenvalues	Relative error present (Bernstein) *n=20*	Relative error present (Bernstein) *n=29*	Relative error Taher *et al* [5] (Cheby-spect-collo)	Relative errors Cheby Gamel and Sameeh [4]	Relative errors Yucel and Boubaker [3] PDQ *N=20*	Relative Errors Yucel and Boubaker [3] PDQ *N=30*
237.72106753	4.697E-012	4.206E-012	2.03E-009	4.697 E-012	7.59E-009	7.59 E-009
2496.48743786	1.269E-012	1.269E-012	7.93E-010	3.046E-012	4.44E-008	4.45E-008
10867.58221698	1.022E-013	1.022E-013	2.33E-010	5.104E-012	1.94E-009	1.71E-008
31780.09645408	3.880E-014	3.388E-014	8.61E-009	8.605E-009	4.50E-008	2.36E-008
74000.84934915	1.994E-012	6.851E-015	7.51E-011		3.97E-005	2.99E-008
148634.47728577	3.053E-009	2.153E-015	2.24E-010		1.43E-004	4.77E-008
269123.43482664	2.262E-013	6.272E-015			4.08E-003	9.61E-010
451247.99471928	1.640E-005	2.966E-015			1.11E-002	1.74E-008
713126.24789600	1.362E-004	1.257E-012			9.02E-002	3.16E-006
1075214.10347396	4.104E-003	2.230E-010			2.06E-002	9.31E-006

Table 2. Comparison of eigenvalues for example1 (a)

Results of Gamel and Sameeh [4] $\lambda_k^{(Cheby)}$	Results of Attili and Lesnic [12]	Results of Abbasbandy and Shirazdi [1]	Results of Syam and Siyyam [13]	Eigenvalue (Bernstein) (present) $\lambda_k^{(Gal.)}$
237.72106753	237.72106753	237.72106753	237.72106754	237.72106753
2496.48743786	2496.48743785	2496.48743785	2496.48743843	2496.48743786
10867.58221704	10867.59367146	10867.58221697	10867.58221699	10867.58221698
31780.09645409	31475.48355038	31780.09645277	31780.09650785	31780.09645408
……………	……………	74000.84934655	74000.85036550	74000. 84934930
……………….	…………….	148634.47747229	148634.47728684	148634. 47773948

Example 1(c): Consider the Sturm-Liouville BVP which is taken from Attili and Lesnic [12]

$$u^{(4)}(x) - \lambda u(x) = 0, \tag{22a}$$

$$u''(0) = u'''(0) = 0, \quad u(1) = u'(1) = 0. \tag{22b}$$

Table 4 shows the comparison of our result obtained using n=22, for Bernstein polynomial, with the first nine eigenvalues of the problem with the results of Attili and Lesnic [12].

Example 2(a): Consider the Sturm-Liouville BVP taken from the articles of Taher *et al* [5] and Attili and Lesnic [12].

$$u^{(4)}(x) = 0.02\,x^2 u''(x) + 0.04\; x u'(x) - (0.0001\,x^4 - 0.02)\,u(x) + \lambda\, u\,(x), \tag{23a}$$

$$\left.\begin{aligned} u(0) = u''(0) = 0 \\ u(5) = u''(5) = 0 \end{aligned}\right\}. \tag{23b}$$

The above problem can be written as self-adjoint form as

$$u^{(4)}(x) - 0.02\left(x^2 u'(x)\right)' + (0.0001\,x^4 - 0.02)u\,(x) = \lambda u\,(x), \tag{24}$$

Table 5 shows the comparison of our result obtained using n=22, for Bernstein polynomial, of the first six eigenvalues of the problem with the results of Yucel and Boubaker [3], Gamel and Sameeh [4], Taher *et al* [5], Attili and Lesnic [12], Syam and Siyyam [13]. The observed CPU time is 5.33 seconds.

Example 2(b): Consider the Sturm-Liouville BVP worked out by Yucel and Boubaker [3], Taher *et al* [5], Attili and Lesnic [12], Chanane [15].

$$u^{(4)}(x) = 0.02\,x^2 u''(x) + 0.04\; x u'(x) - (0.0001\,x^4 - 0.02)\,u(x) + \lambda\, u\,(x), \tag{25a}$$

$$\begin{aligned} &u(0) = u'(0) = 0, \\ &u(5) = u'(5) = 0 \end{aligned} \tag{25b}$$

Table 6 shows the comparison of our result obtained using the degree of polynomial n=22, for Bernstein basis, for the first six eigenvalues of the problem with the results of Yucel and Boubaker, Taher *et al*, Attili and Lesnic, Chanane [3, 5,12, 15] respectively.

Table 3. Comparison of eigenvalues for example 1(b)

$\lambda_k^{(Galerkin)}$ **Bernstein**	**Gamel and Sameeh Cheby-coll. [4]**	**Results of Syam and Siyyam [13]**
500. 563901740	500.563901740	500.563901756
3803. 53708049	3803.53708058	3803.53708049
14617. 6301311	14617.6301777	14617.6301311
39943. 7990057	…………………….	39943.7990057
89135.4076573	………………….	89135.4076571

Table 4. Comparison of eigenvalues for example 1(c)

k	Computed eigenvalue by present method $\lambda_k^{(Gal.)}$	Results of Attili and Lesnic [12]
1	12.3623633683259	12.3623633683262
2	485.518818513372	485.518818513372
3	3806.54626639151	3806.54626639145
4	14617.2733051187	14617.2733051100
5	39943.8317785095	39943.8317790386
6	89135.4050714239	89135.4050444342
7	173881.315656105	173881.315656105
8	308208.452093651	308208.438655408
9	508481.543266068	508481.270992137

Table 5. Comparison of eigenvalues for example 2(a)

Our method *n=22* $\lambda_k^{(galerkin)}$	**Results of Taher et al. [5]**	**Results of Gamel and Sameeh [4]**	**Result of Attili and Lesnic [12]**	**Results of Yucel and Boubaker [3]**	**Results of Syam and Siyyam [13]**
0.21505086437	0.21505086432	0.21505086437	0.2150508643697	0.21505086437	0.21505086437
2.75480993468	2.75480993362	2.7548099346829	2.7548099346829	2.75480993468	2.75480993468
13.21535154056	13.21535154059	13.215351540416	13.215351540558	13.2153515406	13.2153515406
40.95081975916	40.95081975814	40.950820029821	40.950819759137	40.9508197591	40.9508197591
99.05347806349	99.05347803835	………………	99.053478138138	99.0534780633	99.0534781381
204.35573226893	204.35573547934	..……………	204.35449348957	204.355732256	204.3544934895

Table 6. Comparison of eigenvalues for example 2(b)

Our method $\lambda_k^{(galerkin)}$	**Taher et al. [5]**	**Yucel and Boubaker [3]**	**Chanane [15]**	**Attili and Lesnic [12]**
0.86690250239970	0.86690250239196	0.86690250224260	0.86690250239947	0.8669025023997106
6.35768644814590	6.35768644814386	6.35768644843984	6.35768644817446	6.357686448145815
23.99274685030238	23.99274685032633	23.9927468509660	23.99274695066747	23.992746850281375
64.97866759050172	64.97866759484157	64.97866761311830	64.97863591597007	64.97866759571622
144.2806269274497	144.28062688384347	144.2806269273480	....	144.28062803844648
280.6009633049182	280.60096699712966	280.60096374439620		280.58602048195377

5 Conclusion

We have discussed in details the formulations of Sturm-Liouville problem by the Galerkin weighted residual method using Bernstein polynomials as basis functions. To verify the accuracy of our scheme we have considered five examples. In Table 1, we have computed first 10 eigenvalues and compare our results with other published works available in the literature. In Table I, the first seven eigenvalues using Bernstein polynomials are very close to the exact results and the computed values for the lower eigenvalues have a better accuracy than those for the higher eigenvalues. At the same time it is also observed in Table 1 that all 10 eigenvalues, obtained using Bernstein polynomials, converge more rapidly than those obtained by the other methods. In fact relative error decreases as the degree of polynomials increase from n=20 to n=29 in the case of Bernstein basis. But on the other hand, estimated eigenvalues show less convergent especially at present method for n=20. It is obviously observed that eigenvalues obtained by Galerkin-Bernstein method are most accurate than the other results have been achieved by various methods. Excellent agreement is being observed in Table 1 between results of present work and the results of previously published works by Yucel and Boubaker [3], Gamel and Sameeh [4] and Taher et al. [5]. In tables 2, 3 and 5, we have computed 6 eigenvalues and compared our results with Taher et al. [5], Attili and Lesnic [12], Syam and Siyyam [13]. Also using 29 Bernstein polynomials, we obtain the first 9 eigenvalues and compared our results with Attili and Lesnic [8] summarized in Table 4.

The shortcoming of the current method is that, in case of huge number of eigenvalues computation, higher eigenvalues are less convergent than the lower spectrum and with increasing of the degree of polynomials the computational time highly increases, without leading to a significant improvement of the numerical values for some higher order problems. Although slow convergent rate of Bernstein polynomials for some particular problems with complicated boundary conditions makes it less popular still this drawbacks is to be compensated for achieving better accuracy. In spite of these disadvantage, we can conclude that for a relatively small n, i.e., n = 20, moderately precise numerical results are obtained using the proposed method.

Therefore, we may conclude as that Galerkin-Bernstein polynomial scheme produces much accurate results than all other previously published works available in the literature.

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Competing Interests

Authors have declared that no competing interests exist.

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