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Numerical Solution of Stiff and Oscillatory Differential Equations Using a Block Integrator

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Abstract

This paper presents the derivation and implementation of a block integrator for the solution of stiff and oscillatory first-order initial value problems of Ordinary Differential Equations (ODEs). The integrator was derived by collocation and interpolation of the combination of power series and exponential function to generate a continuous implicit Linear Multistep Method (LMM). The basic properties of the derived integrator were investigated and the integrator was implemented on some sampled stiff and oscillatory problems. From the results obtained, it is obvious that the block integrator gives better approximation than some existing ones.

Keywords: Block Integrator, Exponential Function, Oscillatory, Power Series, Stiff.
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1 Introduction

This paper considers the numerical solution of stiff and oscillatory first-order differential equations of the form,

$$y' = f(x, y),\ y(x_0) = y_0,\ x \in [a, b] \tag{1}$$

where x_0 is the initial point, y_0 is the solution at the initial point and f is assumed to satisfy Lipchitz condition stated below.

Theorem 1 [1]: *Let $f(x, y)$ be defined and continuous for all points (x, y) in the region D defined by $a \le x \le b,\ -\infty < y < \infty$, a and b finite, and let there exist a constant L such that, for every x, y, y^* such that (x, y) and (x, y^*) are both in D;*

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$$\left|f(x,y)-f(x,y^{*})\right|\leq L\left|y-y*\right|$$

Then, if y_0 *is any given number, there exists a unique solution* $y(x)$ *of the initial value problem (1), where* $y(x)$ *is continuous and differentiable for all* $(x,y)\, in\, D$.

According to [2], equation (1) is used in simulating the growth of population, trajectory of particles, simple harmonic motion, deflection of a beam, etc. Few equations that are modeled in higher order differential equations are first reduced to systems of first-order before appropriate method of solution is applied. Most often, these problems do not have a closed form solution; hence appropriate methods are adopted to solve such problems. Different methods have been proposed ranging from predictor-corrector methods to block methods. Despite the success recorded by the predictor-corrector method, its major setback is that the predictors are in reducing order of accuracy especially when the value of the step-length is high and moreover the results are at overlapping interval, [3]. Block methods which have advantage of being more efficient in terms of cost implementation, time of execution and accuracy was developed to cater for some of the setbacks of predictor-corrector methods, see [4,5,6,7,8] and [9].

Definition 1 [10]: *A differential equation is said to be stiff if* $\mathrm{Re}(\lambda_i)<0, i=1(1)m$ *, where* λ *is the eigen value of the differential equation*.

Definition 2 [11]: *A nontrivial solution (function) of an ODE is called oscillating if it does not tend either to a finite limit or to infinity (i.e. if it has an infinite number of roots). The differential equation is called oscillating, if it has at least one oscillating solution.*

In search for a method that gives better stability condition, we develop a block integrator for the solution of stiff and oscillatory differential equations using an approximate solution which combines power series with exponential function.

2. Methodology

2.1 Derivation Technique of the Block Integrator

We consider an approximate solution that combines power series and exponential function of the form,

$$y(x)=\sum_{j=0}^{r+s-1}a_j x^j + a_{r+s}\sum_{j=0}^{r+s}\frac{\alpha^j x^j}{j!} \qquad (2)$$

Interpolation and collocation procedures are used by choosing interpolation point s at a grid point and collocation points r at all points giving rise to $\xi = s+r$ system of equations whose coefficients are determined by using appropriate procedures. The first derivative of (2) is given by,

$$y'(x) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=1}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (3)$$

where $a_j, \alpha^j \in \Re$ for $j = 0(1)7$ and $y(x)$ is continuously differentiable. Let the solution of (1) be sought on the partition $\pi_N : a = x_0 < x_1 < x_2 < \ldots < x_n < x_{n+1} < \ldots < x_N = b$, of the integration interval $[a,b]$ with a constant step-size h, given by, $h = x_{n+1} - x_n$, $n = 0,1,\ldots,N$.

Then, substituting (3) in (1) gives,

$$f(x,y) = \sum_{j=0}^{r+s-1} j a_j x^{j-1} + a_{r+s} \sum_{j=1}^{r+s} \frac{\alpha^j x^{j-1}}{(j-1)!} \qquad (4)$$

Now, interpolating (2) at point $x_{n+s}, s = 0$ and collocating (4) at points $x_{n+r}, r = 0(1)6$, leads to the following system of equations,

$$AX = U \qquad (5)$$

where

$$A = [a_0 \;\; a_1 \;\; a_2 \;\; a_3 \;\; a_4 \;\; a_5 \;\; a_6 \;\; a_7]^T$$
$$U = [y_n \;\; f_n \;\; f_{n+1} \;\; f_{n+2} \;\; f_{n+3} \;\; f_{n+4} \;\; f_{n+5} \;\; f_{n+6}]^T$$

and

$$X = \begin{bmatrix}
1 & x_n & x_n^2 & x_n^3 & x_n^4 & x_n^5 & x_n^6 & \left(1 + \alpha x_n + \frac{\alpha^2 x_n^2}{2!} + \frac{\alpha^3 x_n^3}{3!} + \frac{\alpha^4 x_n^4}{4!} + \frac{\alpha^5 x_n^5}{5!} + \frac{\alpha^6 x_n^6}{6!} + \frac{\alpha^7 x_n^7}{7!}\right) \\
0 & 1 & 2x_n & 3x_n^2 & 4x_n^3 & 5x_n^4 & 6x_n^5 & \left(\alpha + \alpha^2 x_n + \frac{\alpha^3 x_n^2}{2!} + \frac{\alpha^4 x_n^3}{3!} + \frac{\alpha^5 x_n^4}{4!} + \frac{\alpha^6 x_n^5}{5!} + \frac{\alpha^7 x_n^6}{6!}\right) \\
0 & 1 & 2x_{n+1} & 3x_{n+1}^2 & 4x_{n+1}^3 & 5x_{n+1}^4 & 6x_{n+1}^5 & \left(\alpha + \alpha^2 x_{n+1} + \frac{\alpha^3 x_{n+1}^2}{2!} + \frac{\alpha^4 x_{n+1}^3}{3!} + \frac{\alpha^5 x_{n+1}^4}{4!} + \frac{\alpha^6 x_{n+1}^5}{5!} + \frac{\alpha^7 x_{n+1}^6}{6!}\right) \\
0 & 1 & 2x_{n+2} & 3x_{n+2}^2 & 4x_{n+2}^3 & 5x_{n+2}^4 & 6x_{n+2}^5 & \left(\alpha + \alpha^2 x_{n+2} + \frac{\alpha^3 x_{n+2}^2}{2!} + \frac{\alpha^4 x_{n+2}^3}{3!} + \frac{\alpha^5 x_{n+2}^4}{4!} + \frac{\alpha^6 x_{n+2}^5}{5!} + \frac{\alpha^7 x_{n+2}^6}{6!}\right) \\
0 & 1 & 2x_{n+3} & 3x_{n+3}^2 & 4x_{n+3}^3 & 5x_{n+3}^4 & 6x_{n+3}^5 & \left(\alpha + \alpha^2 x_{n+3} + \frac{\alpha^3 x_{n+3}^2}{2!} + \frac{\alpha^4 x_{n+3}^3}{3!} + \frac{\alpha^5 x_{n+3}^4}{4!} + \frac{\alpha^6 x_{n+3}^5}{5!} + \frac{\alpha^7 x_{n+3}^6}{6!}\right) \\
0 & 1 & 2x_{n+4} & 3x_{n+4}^2 & 4x_{n+4}^3 & 5x_{n+4}^4 & 6x_{n+4}^5 & \left(\alpha + \alpha^2 x_{n+4} + \frac{\alpha^3 x_{n+4}^2}{2!} + \frac{\alpha^4 x_{n+4}^3}{3!} + \frac{\alpha^5 x_{n+4}^4}{4!} + \frac{\alpha^6 x_{n+4}^5}{5!} + \frac{\alpha^7 x_{n+4}^6}{6!}\right) \\
0 & 1 & 2x_{n+5} & 3x_{n+5}^2 & 4x_{n+5}^3 & 5x_{n+5}^4 & 6x_{n+5}^5 & \left(\alpha + \alpha^2 x_{n+5} + \frac{\alpha^3 x_{n+5}^2}{2!} + \frac{\alpha^4 x_{n+5}^3}{3!} + \frac{\alpha^5 x_{n+5}^4}{4!} + \frac{\alpha^6 x_{n+5}^5}{5!} + \frac{\alpha^7 x_{n+5}^6}{6!}\right) \\
0 & 1 & 2x_{n+6} & 3x_{n+6}^2 & 4x_{n+6}^3 & 5x_{n+6}^4 & 6x_{n+6}^5 & \left(\alpha + \alpha^2 x_{n+6} + \frac{\alpha^3 x_{n+6}^2}{2!} + \frac{\alpha^4 x_{n+6}^3}{3!} + \frac{\alpha^5 x_{n+6}^4}{4!} + \frac{\alpha^6 x_{n+6}^5}{5!} + \frac{\alpha^7 x_{n+6}^6}{6!}\right)
\end{bmatrix}$$

Solving (5), for a_j's, $j = 0(1)7$ and substituting back into (2) gives a continuous linear multistep method of the form:

$$y(x) = \alpha_0(x) y_n + h \sum_{j=0}^{6} \beta_j(x) f_{n+j} \qquad (6)$$

where the coefficients of y_n and f_{n+j} are given by,

$$\left.\begin{aligned}
&\alpha_0 = 1\\
&\beta_0 = \frac{1}{60480}(12t^7 - 294t^6 + 2940t^5 - 15435t^4 + 45472t^3 - 74088t^2 + 60480t)\\
&\beta_1 = -\frac{1}{2520}(3t^7 - 70t^6 + 651t^5 - 3045t^4 + 7308t^3 - 7560t^2)\\
&\beta_2 = \frac{1}{20160}(60t^7 - 1330t^6 + 11508t^5 - 48405t^4 + 98280t^3 - 75600t^2)\\
&\beta_3 = -\frac{1}{3780}(15t^7 - 315t^6 + 2541t^5 - 9765t^4 + 17780t^3 - 12600t^2)\\
&\beta_4 = \frac{1}{20160}(60t^7 - 1190t^6 + 8988t^5 - 32235t^4 + 55440t^3 - 37800t^2)\\
&\beta_5 = -\frac{1}{2520}(3t^7 - 56t^6 + 399t^5 - 1365t^4 + 2268t^3 - 1512t^2)\\
&\beta_6 = \frac{1}{60480}(12t^7 - 210t^6 + 1428t^5 - 4725t^4 + 7672t^3 - 5040t^2)
\end{aligned}\right\} \tag{7}$$

where $t = (x - x_n)/h$. Evaluating (6) at $t = 1(1)6$ gives a block scheme of the form:

$$A^{(0)}\mathbf{Y}_m = \mathbf{E}\mathbf{y}_n + hd\mathbf{f}(\mathbf{y}_n) + hb\mathbf{F}(\mathbf{Y}_m) \tag{8}$$

where $\mathbf{Y}_m = \left[y_{n+1}\ \ y_{n+2}\ \ y_{n+3}\ \ y_{n+4}\ \ y_{n+5}\ \ y_{n+6}\right]^T$, $\mathbf{y}_n = \left[y_{n-5}\ \ y_{n-4}\ \ y_{n-3}\ \ y_{n-2}\ \ y_{n-1}\ \ y_n\right]^T$

$\mathbf{F}(\mathbf{Y}_m) = \left[f_{n+1}\ \ f_{n+2}\ \ f_{n+3}\ \ f_{n+4}\ \ f_{n+5}\ \ f_{n+6}\right]^T$, $\mathbf{f}(\mathbf{y}_n) = \left[f_{n-5}\ \ f_{n-4}\ \ f_{n-3}\ \ f_{n-2}\ \ f_{n-1}\ \ f_n\right]^T$

$$A^{(0)} = \begin{bmatrix} 1&0&0&0&0&0\\ 0&1&0&0&0&0\\ 0&0&1&0&0&0\\ 0&0&0&1&0&0\\ 0&0&0&0&1&0\\ 0&0&0&0&0&1 \end{bmatrix},\quad \mathbf{E} = \begin{bmatrix} 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1\\ 0&0&0&0&0&1 \end{bmatrix},\quad d = \begin{bmatrix} 0&0&0&0&0&\frac{19087}{60480}\\ 0&0&0&0&0&\frac{1139}{3780}\\ 0&0&0&0&0&\frac{137}{448}\\ 0&0&0&0&0&\frac{286}{945}\\ 0&0&0&0&0&\frac{3715}{12096}\\ 0&0&0&0&0&\frac{41}{140} \end{bmatrix}$$

$$b=\begin{bmatrix}\frac{2713}{2520} & \frac{-15487}{20160} & \frac{586}{945} & \frac{-6737}{20160} & \frac{263}{2520} & \frac{-863}{60480}\\ \frac{94}{63} & \frac{11}{1260} & \frac{332}{945} & \frac{-269}{1260} & \frac{22}{315} & \frac{-37}{3780}\\ \frac{81}{56} & \frac{1161}{2240} & \frac{34}{35} & \frac{-729}{2240} & \frac{27}{280} & \frac{-29}{2240}\\ \frac{464}{315} & \frac{128}{315} & \frac{1504}{945} & \frac{58}{315} & \frac{16}{315} & \frac{-8}{945}\\ \frac{725}{504} & \frac{2125}{4032} & \frac{250}{189} & \frac{3875}{4032} & \frac{235}{504} & \frac{-275}{12096}\\ \frac{54}{35} & \frac{27}{140} & \frac{68}{35} & \frac{27}{140} & \frac{54}{35} & \frac{41}{140}\end{bmatrix}$$

3. Analysis of Basic Properties of the Block Integrator

3.1 Order of the Block Integrator

Let the linear operator $L\{y(x);h\}$ associated with the block (8) be defined as,

$$L\{y(x);h\}=A^{(0)}Y_m-Ey_n-hdf(y_n)-hbF(Y_m) \tag{9}$$

Expanding (9) using Taylor series and comparing the coefficients of h gives,

$$L\{y(x);h\}=c_0y(x)+c_1hy'(x)+c_2h^2y''(x)+...+c_ph^py^p(x)+c_{p+1}h^{p+1}y^{p+1}(x)+... \tag{10}$$

Definition 3 [12]:The linear operator L and the associated continuous linear multistep method (6) are said to be of order p if $c_0=c_1=c_2=...=c_p=0$ *and* $c_{p+1}\neq 0$. c_{p+1} is called the error constant and the local truncation error is given by,

$$t_{n+k}=c_{p+1}h^{(p+1)}y^{(p+1)}(x_n)+O(h^{p+2}) \tag{11}$$

For our block integrator,

$$L\{y(x);h\}=\begin{bmatrix}1&0&0&0&0&0\\0&1&0&0&0&0\\0&0&1&0&0&0\\0&0&0&1&0&0\\0&0&0&0&1&0\\0&0&0&0&0&1\end{bmatrix}\begin{bmatrix}y_{n+1}\\y_{n+2}\\y_{n+3}\\y_{n+4}\\y_{n+5}\\y_{n+6}\end{bmatrix}-\begin{bmatrix}1\\1\\1\\1\\1\\1\end{bmatrix}[y_n]-h\begin{bmatrix}\frac{19087}{60480} & \frac{2713}{2520} & \frac{-15487}{20160} & \frac{586}{945} & \frac{-6737}{20160} & \frac{263}{2520} & \frac{-863}{60480}\\ \frac{1139}{3780} & \frac{94}{63} & \frac{11}{1260} & \frac{332}{945} & \frac{-269}{1260} & \frac{22}{315} & \frac{-37}{3780}\\ \frac{137}{448} & \frac{81}{56} & \frac{1161}{2240} & \frac{34}{35} & \frac{-729}{2240} & \frac{27}{280} & \frac{-29}{2240}\\ \frac{286}{945} & \frac{464}{315} & \frac{128}{315} & \frac{1504}{945} & \frac{58}{315} & \frac{16}{315} & \frac{-8}{945}\\ \frac{3715}{12096} & \frac{725}{504} & \frac{2125}{4032} & \frac{250}{189} & \frac{3875}{4032} & \frac{235}{504} & \frac{-275}{12096}\\ \frac{41}{140} & \frac{54}{35} & \frac{27}{140} & \frac{68}{35} & \frac{27}{140} & \frac{54}{35} & \frac{41}{140}\end{bmatrix}\begin{bmatrix}f_n\\f_{n+1}\\f_{n+2}\\f_{n+3}\\f_{n+4}\\f_{n+5}\\f_{n+6}\end{bmatrix}=0 \tag{12}$$

Expanding (12) in Taylor series gives:

$$\begin{bmatrix} \sum_{j=0}^{\infty}\frac{(h)^j}{j!}y_n^j - y_n - \frac{19087h}{60480}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{2713}{2520}(1)^j - \frac{15487}{20160}(2)^j + \frac{586}{945}(3)^j - \frac{6737}{20160}(4)^j + \frac{263}{2520}(5)^j - \frac{863}{60480}(6)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(2h)^j}{j!}y_n^j - y_n - \frac{1139h}{3780}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{94}{63}(1)^j + \frac{11}{1260}(2)^j + \frac{332}{945}(3)^j - \frac{269}{1260}(4)^j + \frac{22}{315}(5)^j - \frac{37}{3780}(6)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(3h)^j}{j!}y_n^j - y_n - \frac{137h}{448}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{81}{56}(1)^j + \frac{11}{2240}(2)^j + \frac{34}{35}(3)^j - \frac{729}{2240}(4)^j + \frac{27}{280}(5)^j - \frac{29}{2240}(6)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(4h)^j}{j!}y_n^j - y_n - \frac{286h}{945}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{464}{315}(1)^j + \frac{128}{315}(2)^j + \frac{1504}{945}(3)^j + \frac{58}{315}(4)^j + \frac{16}{315}(5)^j - \frac{8}{945}(6)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(5h)^j}{j!}y_n^j - y_n - \frac{315h}{12096}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{725}{504}(1)^j + \frac{2125}{4032}(2)^j + \frac{250}{189}(3)^j + \frac{3875}{4032}(4)^j + \frac{235}{504}(5)^j - \frac{275}{12096}(6)^j\right\} \\ \sum_{j=0}^{\infty}\frac{(6h)^j}{j!}y_n^j - y_n - \frac{41h}{140}y_n' - \sum_{j=0}^{\infty}\frac{h^{j+1}}{j!}y_n^{j+1}\left\{\frac{54}{35}(1)^j + \frac{27}{140}(2)^j + \frac{68}{35}(3)^j + \frac{27}{140}(4)^j + \frac{54}{35}(5)^j + \frac{41}{140}(6)^j\right\} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \tag{13}$$

Hence,

$$\bar{c}_0 = \bar{c}_1 = \bar{c}_2 = \bar{c}_3 = \bar{c}_4 = \bar{c}_5 = \bar{c}_6 = \bar{c}_7 = 0,$$
$$\bar{c}_8 = \left[0.010(-03)\ \ 0.006(-03)\ \ 0.08(-03)\ \ 0.006(-03)\ \ 0.009(-03)\ \ -0.001(-03)\right]^T$$

Therefore, the block integrator is of order seven.

3.2 Zero Stability

Definition 4 [12]: *The block integrator (8) is said to be zero-stable, if the roots* $z_s, s = 1, 2, \ldots, k$ *of the first characteristic polynomial* $\rho(z)$ *defined by* $\rho(z) = \det(z\mathbf{A}^{(0)} - \mathbf{E})$ *satisfies* $|z_s| \le 1$ *and every root satisfying* $|z_s| \le 1$ *have multiplicity not exceeding the order of the differential equation. Moreover, as* $h \to 0$, $\rho(z) = z^{r-\mu}(z-1)^{\mu}$ *where* μ *is the order of the differential equation,* r *is the order of the matrices* $\mathbf{A}^{(0)}$ *and* $\mathbf{E}$, see [13] for details.

For our block integrator,

$$\rho(z) = \left| z\begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} - \begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \right| = 0 \tag{14}$$

$\rho(z) = z^5(z-1) = 0, \Rightarrow z_1 = z_2 = z_3 = z_4 = z_5 = 0, z_6 = 1$. Hence, the block integrator is zero-stable.

3.3 Consistency

The block integrator (8) is consistent since it has order $p = 7 \geq 1$.

3.4 Convergence

The block integrator is convergent by consequence of Dahlquist theorem below.

Theorem 2 [14]: *The necessary and sufficient conditions that a continuous LMM be convergent are that it be consistent and zero-stable.*

3.5 Region of Absolute Stability

Definition 5 [15]***:*** *Region of absolute stability is a region in the complex* z *plane, where* $z = \lambda h$. *It is defined as those values of* z *such that the numerical solutions of* $y' = -\lambda y$ *satisfy* $y_j \to 0$ *as* $j \to \infty$ *for any initial condition.*

We shall adopt the boundary locus method to determine the region of absolute stability of the block integrator. This is achieved by substituting the test equation,

$$y' = -\lambda y \tag{15}$$

into the block formula gives (8). This gives,

$$\mathbf{A}^{(0)}\mathbf{Y}_m(w) = \mathbf{E}\mathrm{y}_n(w) - h\lambda\mathbf{D}y_n(w) - h\lambda\mathbf{B}\mathbf{Y}_m(w) \tag{16}$$

Thus,

$$\bar{h}(w) = -\left(\frac{\mathbf{A}^{(0)}Y_m(w) - \mathbf{E}y_n(w)}{\mathbf{D}y_n(w) + \mathbf{B}Y_m(w)}\right) \tag{17}$$

since $\bar{h}$ is given by $\bar{h} = \lambda h$ *and* $w = e^{i\theta}$. Equation (17) is our characteristic/stability polynomial. For the block integrator, equation (17) is given by,

$$\begin{aligned}\bar{h}(w) = &-h^6\left(\frac{1}{7}w^5 - \frac{1}{7}w^6\right) - h^5\left(\frac{7}{10}w^6 + \frac{7}{10}w^5\right) - h^4\left(\frac{29}{15}w^5 - \frac{29}{15}w^6\right) \\ &- h^3\left(\frac{7}{2}w^6 + \frac{7}{2}w^5\right) - h^2\left(\frac{25}{6}w^5 - \frac{25}{6}w^6\right) - h\left(3w^6 + 3w^5\right) + w^6 - w^5\end{aligned} \tag{18}$$

This gives the stability region shown in the figure below.

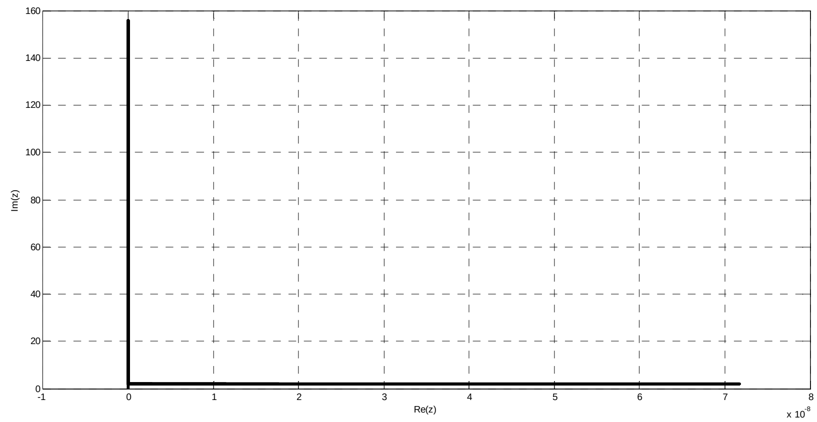

Fig. 1. Showing Region of Absolute Stability of the Block Integrator

According to [12], stiff algorithms have unbounded RAS. Thus, from Fig. 1 above, the extended block integrator is suitable for solving stiff problems. Also, [10] said that the stability region for L-stable schemes must encroach into the positive half of the complex z plane. Thus, the block integrator is L-stable.

4. Numerical Experiments

We shall evaluate the performance of the block integrator on some challenging stiff and oscillatory problems which have appeared in literature and compare the results with solutions from some methods of similar derivation. The following notations shall be used in the tables below;

ERR- |Exact Solution-Computed Solution|
ERO- Error in [16]
ERA- Error in [17]
ERS- Error in [18]

4.1 Numerical Examples

Problem 1:

Consider the highly stiff ODE

$$y' = -10(y-1)^2,\ y(0) = 2 \tag{19}$$

which has the exact solution,

$$y(x) = 1 + \frac{1}{1+10x} \tag{20}$$

This problem was earlier discussed by [10], he showed that many predictor-corrector and block methods become unstable with this problem, including the hybrid methods. However, the newly derived block integrator is used for the integration of this problem within the interval $0 \leq x \leq 0.1$. Authors in [16] solved this stiff problem by adopting a new 2-point block method with step size ratio at $r = 1$. Authors in [18] also solved problem 1 by applying a self-starting block integrator.

Problem 2:

Consider the Prothero-Robinson oscillatory ODE,

$$y' = L(y - \sin x) + \cos x,\ L = -1,\ y(0) = 0 \tag{21}$$

with the exact solution,

$$y(x) = \sin x \tag{22}$$

Authors in [17] solved this problem by adopting a generalized rational approximation method via Pade approximants with step number $k = 6\,.\,r = 1$. Authors in [18] also solved problem 2 by applying a self-starting block integrator.

Table 1. Showing the results for stiff problem 1

x	Exact solution	Computed solution	ERR	ERS	ERO
0.0100	1.9090909090909092	1.9090909868074991	5.222834e-008	3.414671e-006	1.07e-03
0.0200	1.8333333333333335	1.8333334606188648	8.727145e-008	2.749635e-006	2.38e-03
0.0300	1.7692307692307692	1.7692307604778971	1.069875e-008	1.342943e-006	2.21e-03
0.0400	1.7142857142857144	1.7142857127875963	8.987150e-008	9.090648e-006	5.36e-03
0.0500	1.6666666666666665	1.6666666243797619	4.712423e-008	7.969685e-006	7.53e-03
0.0600	1.6250000000000000	1.6250000175525943	1.808182e-008	6.994886e-006	9.00e-03
0.0700	1.5882352941176470	1.5882352922979736	1.602002e-008	6.270048e-006	9.98e-03
0.0800	1.5555555555555556	1.5555555882520038	1.429167e-008	6.017101e-006	1.06e-02
0.0900	1.5263157894736841	1.5263157601947504	1.283029e-008	5.411308e-006	1.10e-02
0.1000	1.5000000000000000	1.4999999213542157	1.159479e-008	4.880978e-006	1.12e-02

Table 2. Showing the Results for Prothero-Robinson Oscillatory Problem 2

x	Exact solution	Computed solution	ERR	ERS	ERA
0.1000	0.0998334166468282	0.0998334166468182	1.822016e-014	3.703180e-012	2.0e-11
0.2000	0.1986693307950612	0.1986693307950227	2.271482e-014	6.102036e-012	3.0e-11
0.3000	0.2955202066613396	0.2955202066613424	4.241108e-014	1.733789e-012	1.0e-10
0.4000	0.3894183423086505	0.3894183423086136	1.364169e-014	1.115490e-012	2.0e-10
0.5000	0.4794255386042030	0.4794255386042274	6.502551e-014	2.226122e-011	1.0e-10
0.6000	0.5646424733950355	0.5646424733950910	9.103963e-014	5.567768e-012	2.0e-10
0.7000	0.6442176872376911	0.6442176872376195	1.951339e-014	7.511613e-012	1.0e-10
0.8000	0.7173560908995228	0.7173560908995716	7.155093e-014	1.253389e-011	2.0e-10
0.9000	0.7833269096274835	0.7833269096274592	5.921081e-014	1.501860e-012	3.0e-10
1.0000	0.8414709848078966	0.8414709848078846	8.457038e-014	1.803588e-011	3.0e-10

4.2 Discussion of Results

We have considered two numerical examples in this paper. The first problem (which is stiff) was solved by authors in [16] where they applied 2-point block method with step-size ratio at $r=1$ while the second problem (which is oscillatory) was solved by authors in [17] where they adopted generalized rational approximation method via Pade approximants with step number $k=6$. We solved the two problems using the new block integrator developed. Tables 1 and 2 above showed that the block integrator gives better results than the existing ones.

4. Conclusion

We have presented a block integrator for the solution of stiff and oscillatory first-order ordinary differential equations. Our aim was to construct highly stable block integrator which is computationally more efficient than many of the existing numerical integrators for stiff and oscillatory problems. The approximate solution (basis function) adopted in this paper produced a block integrator with L-stable stability region. This made it possible for the block integrator to perform well on stiff and oscillatory problems. The block integrator proposed was found to be zero-stable, consistent and convergent. The block integrator was also found to perform better than some existing methods in view of the numerical results obtained.

Competing Interests

The authors declare that they have no competing interests

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