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The Existence Result for a Fractional Kirchhoff Equation Involving Doubly Critical Exponents and Combined Nonlinearities

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Abstract

This paper zeroes in on the existence result of solutions to a fractional Kirchhoff equation with doubly critical exponents, mixed nonlinear terms and a continuous potential *V*. After utilizing some energy estimates, one obtains the effect of exponents p and q on the existence of constrained minimizers, namely, the connection between the existence of normalized solutions and exponents *p*, *q*.

Subject Areas

Functional Analysis

Keywords

Fractional Kirchhoff Equations, Constrained Minimizers, Doubly Critical Exponents

1. Introduction

This paper is focused on the fractional Kirchhoff equation with combined nonlinearities as follows:

$$\left(a+b\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x\right)(-\Delta)^s u+V(x)u=\lambda u+c|u|^{p-2}u+d|u|^{q-2}u \text{ in } \mathbb{R}^N \quad (1.1)$$

where a is a positive constant, which will be defined specifically in the sequel, $b>0$, $1\le N\le 3$, $0<s<1$, $2<q<p=4=2_s^*$ and $c>0$, $d\ne 0$, λ is a Lagrange constant. The fractional Laplacion $(-\Delta)^s\,(s\in(0,1))$ can be defined as

$$(-\Delta)^s v(x)=C_s \mathrm{P.V.}\int_{\mathbb{R}^N}\frac{v(x)-v(y)}{|x-y|^{N+2s}}\mathrm{d}y=C_s\lim_{\varepsilon\to 0}\int_{\mathbb{R}^N\setminus B_\varepsilon(x)}\frac{v(x)-v(y)}{|x-y|^{N+2s}}\mathrm{d}y$$

for $v \in S(\mathbb{R}^N)$, where $S(\mathbb{R}^N)$ is the Schwartz space of rapidly decaying C^∞ function, $B_\varepsilon(x)$ denotes an open ball of radius ε centered at $x \in \mathbb{R}^N$ and the constant $C_s = \left(\int_{\mathbb{R}^N} \frac{1-\cos(\xi_1)}{|\xi|^{N+2s}} \right)^{-1}$.

For the case of $a > 0$, $b > 0$, $s = 1$, problem (1.1) is a classical Kirchhoff equation. And this type of equation has been associated with the following equation

$$u_{tt} - \left(a + b\int_{\mathbb{R}^N} |\nabla u|^2 \,\mathrm{d}x \right) \Delta u = f(x,u) \quad \text{in } \Omega \times (0,\infty). \tag{1.2}$$

Problem 1.2 was proposed by Kirchhoff [1] in 1883 at the outset, where he obtained the classical D'Alembert wave equation, where the nonlinearity $f(x,u)$ is of general type. Besides, the physical and biological background of (1.2) can be found in [2] [3] and the references therein. And it has brought itself into notice after the seminal contribution of [4]. Next let's study this type of equation:

$$\left(a + b\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} u \right|^2 \mathrm{d}x \right) (-\Delta)^s u - \lambda u = f(x,u) \text{ in } \mathbb{R}^N. \tag{1.3}$$

When $s = 1$, equation (1.3) is a type of typical Kirchhoff equation. And in the recent years, it has been studied by many authors. For example, in [5], He and Zou obtained the existence and concentration behavior of positive solutions for a Kirchhoff equation. In [6], Figueiredo *et al.* studied the existence and concentration results for a Kirchhoff type equation with general nonlinearities. For more results about the existence of solutions to the Kirchhoff type equation like (1.3) with $s = 1$, we refer readers to [7] [8] and the references therein.

Moreover, for the case of $0 < s < 1$, namely, for the nonlocal operator $(-\Delta)^s$, its background can be found in several areas such as fractional quantum mechanics [9], physics [10] and so on. About the fractional Kirchhoff problems, to the best of our knowledge, a lot of authors have obtained fruitful results. For example, in [11], Caffarelli and Silvestre introduced the harmonic extension method changing this nonlocal problem into a local one in higher dimensions. In [12], Gu and Yang studied a singular perturbation fractional Kirchhoff equation in the critical case. Furthermore, readers can refer to [13] [14] and the references therein for more results on the existence of solutions for the fractional Kirchhoff equation (1.3).

Motivated by Li and Chen ([15] and [16]), the aim of this paper is to generalize their results to the case of mixed nonlinearities.

By direct computation, it is easy to find that if $N = 4s$, the critical Sobolev exponent $2_s^* = \frac{2N}{N-2s}$ and the fractional Gagliardo-Nirenberg-Sobolev critical exponent $2_{GNS}^* = 2 + \frac{8s}{N}$ are equal, moreover, $\frac{2N}{N-2s} = \frac{2N+8s}{N} = 4$. And in this paper, we study the case of $N = 4s$, namely, the doubly critical exponents case. Besides, we have

$$\begin{cases} \text{if } N=1, \text{ then } s=\dfrac{1}{4}, \\ \text{if } N=2, \text{ then } s=\dfrac{1}{2}, \\ \text{if } N=3, \text{ then } s=\dfrac{3}{4}. \end{cases} \tag{1.4}$$

It is customary that a (weak) solution of problem (1.1) is a critical point of the energy functional

$$E_V^c(u)=\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x+\frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^2$$
$$-\frac{c}{4}\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x-\frac{d}{q}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x,$$

constrained on

$$S_V:=\left\{u\in H_V^s:\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x=1\right\},$$

where

$$H_V^s:=\left\{u\in H^s\left(\mathbb{R}^N\right):\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x<\infty\right\}.$$

The fractional Sobolev space $H^s\left(\mathbb{R}^N\right)$ is defined as

$$H^s\left(\mathbb{R}^N\right):=\left\{u\in L^2\left(\mathbb{R}^N\right):\frac{u(x)-u(y)}{|x-y|^{\frac{N+2s}{2}}}\in L^2\left(\mathbb{R}^N\times\mathbb{R}^N\right)\right\}$$

with the norm

$$\|u\|_{H^s\left(\mathbb{R}^N\right)}=\left(\int_{\mathbb{R}^N}\left(\left|(-\Delta)^{\frac{s}{2}}u\right|^2+|u|^2\right)\mathrm{d}x\right)^{\frac{1}{2}},$$

where

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x=\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.$$

By remark 1.5.1 of [17], we know the fact that a (local) point of minimum of a differentiable functional is a critical point. Then we study the minimization problem with respect to the fractional Kirchhoff functional on the L^2-constrained manifold:

$$m_V(c):=\inf_{u\in S_V}E_V^c(u). \tag{1.5}$$

This paper $|\cdot|_p$ denotes the norm of $L^p\left(\mathbb{R}^N\right)$ defined by $|u|_p^p=\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x$. If $V=0$, we denote the space H_V^s by H_0^s, the set S_V by S_0, the functional E_V^c by E_0^c, and $m_V(c)$ by $m_0(c)$ respectively.

By the above notations, we are ready to give the main result of this paper, namely, a result about the minimization problem (1.5).

Theorem 1.1. *Let* $2<q<p=4$, $a\geq \frac{2dS_s^{2-q}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2(q-2)}-\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x}{\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2}$ and $a>0$ (*where* S_s *is defined in Lemma 2.1*), $c>0$, $d>0$, *and* $V(\cdot)$ *satisfies the following condition*:

(C) $V\in C\left(\mathbb{R}^N,[0,\infty)\right)$, $\lim_{|x|\to\infty}V(x)=\infty$, $\inf_{x\in\mathbb{R}^N}V(x)=0$, *and there exists a sufficiently small* $\varepsilon_0>0$ *such that meas* $\left(\{V(x)\leq\varepsilon_0\}\right)\leq\varepsilon_0$.

Then, there exists a positive constant c_* *such that if* $2<q<p=4$, *then* $c_*=bS_s^2$ *and*

$$\begin{cases} m_V(c)\in(-\infty,0), & \text{if } c\in(0,c_*),\\ m_V(c)=0, & \text{if } c=c_*,\\ m_V(c)=-\infty, & \text{if } c\in(c_*,\infty).\end{cases}$$

Furthermore, $E_V^c(u)$ *has an energy minimizer for* $c<c_*$, *and has no minimizers for* $c\geq c_*$.

Remark 1.1. *By Theorem 1.1, we get a threshold value of* $c>0$ *which separates the existence and nonexistence of minimizers, which improves* ([15], *Theorem 1.2*), *where the existence of minimizers for* (*1.1*) *with* $s=1$ *and* $d=0$ *is obtained. The main obstruction is to impose energy estimates to characterize the threshold value* c_* *and the infimum energy level* $m_V(c)$ *for* $2<q<p=4$.

Remark 1.2. *Theorem 1.1 is also true in the case of* $d<0$, *with only a small change in the case of* $d>0$, *which we omit here.*

Remark 1.3. *For* $V(x)=|x|^2$, *readers can verify that it satisfies the condition* (*C*) *in Theorem 1.1.*

This paper is organized as follows: We first list some preliminaries in Section 2, the main proof of Theorem 1.1 will be given in Section 3 and finally, we summarize the main contents of this paper in Section 4.

2. Preliminaries

In this section, some results which will be used frequently throughout the rest of the paper are firstly listed below.

Lemma 2.1. ([18]) *Let* $s\in(0,1)$ *and* $p\in[1,+\infty)$ *be such that* $sp<N$. *Then, there exists a positive constant* $S_s=S_s(N,p,s)$ *such that, for any measurable and compactly supported function* $u:\mathbb{R}^N\to\mathbb{R}$, *one has*

$$S_s|u|_{2_s^*}^2\leq\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y, \tag{2.1}$$

where $2_s^*=\frac{2N}{N-2s}$ *is the so-called fractional critical Sobolev exponent. Moreover, equality* (*2.2*) *holds if and only if* $\tilde{u}=\mathcal{K}\left(\mu^2+|x-x_0|^2\right)^{-\frac{N-2s}{2}}$ *with*

$\mathcal{K} \in \mathbb{R}\setminus\{0\}$, $\mu > 0$, $x_0 \in \mathbb{R}^N$ *fixed constants,* S_s *is the best Sobolev embedding constant.*

Lemma 2.2 ([19]) *Let* $p \in \left[0, 2_s^* - 2\right)$ *and* $u \in H^s\left(\mathbb{R}^N\right)$, *then inequality*

$$\int_{\mathbb{R}^N} |u|^{p+2}\,\mathrm{d}x \le \frac{p+2}{|Q|_2^p}\alpha_p\beta_p\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^{\frac{Np}{4s}}\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{2ps-Np+4s}{4s}} \tag{2.2}$$

holds, where $\alpha_p = \dfrac{2s}{2ps - Np + 4s}$, $\beta_p = \left(\dfrac{2ps - Np + 4s}{Np}\right)^{\frac{Np}{4s}}$ *and the function* $Q(x)$ *optimizes* (*2.2*) *and is the unique nonnegative radically solution of the fractional nonlinear equation*

$$(-\Delta)^s Q + Q - |Q|^p Q = 0 \quad \text{in } \mathbb{R}^N.$$

According to Lemma 2.1 and Lemma 2.2, when $N = 4s$ and $|u|_2 = 1$, the fractional Sobolev inequality (2.1) and the Gagliardo-Nirenberg-Sobolev inequality (2.2) can be rewritten, in other words, the fractional Sobolev inequality (2.1) turns into

$$S_s^2\int_{\mathbb{R}^N}|u|^4\,\mathrm{d}x \le \left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^2, \quad u \in H^s\left(\mathbb{R}^N\right). \tag{2.3}$$

If we change p into $p-2$ in (2.2), then the fractional Gagliardo-Nirenberg-Sobolev inequality (2.2) becomes

$$\int_{\mathbb{R}^N}|u|^p\,\mathrm{d}x \le \frac{p}{|Q|_2^{p-2}}\alpha_p\beta_p\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^{p-2}, \quad u \in H^s\left(\mathbb{R}^N\right), \tag{2.4}$$

with the equality holds when $u = \lambda^{\frac{N}{2}}Q(\lambda x)/|Q|_2$, where $\alpha_p = \dfrac{1}{4-p}$ and $\beta_p = \left(\dfrac{4-p}{2(p-2)}\right)^{p-2}$.

Particularly, when $U = \bar{u}\left(\dfrac{x}{S_s^{\frac{1}{2s}}}\right)$, where $\bar{u} = \dfrac{\tilde{u}}{|\tilde{u}|_{2_s^*}}$, one has

$$S_s^2 = \frac{\left|(-\Delta)^{\frac{s}{2}}U\right|_2^4}{|U|_4^4}, S_s^2 = \left|(-\Delta)^{\frac{s}{2}}U\right|_2^2 = |U|_4^4.$$

Similar to ([20], Lemma 5.1), one gets the result about the embedding as follows:

Lemma 2.3. *Assume that* $V(x)$ *satisfies condition* (*C*), *then the embedding* $H_V^s \hookrightarrow L^p\left(\mathbb{R}^N\right)$ *is compact for* $p \in [2,4)$.

The proof of this lemma has already been given in ([16], Lemma 2.3), but for the readers' convenience, we sketch it here again.

Proof. **Step 1**. We first show that $H_V^s \hookrightarrow L^p(\mathbb{R}^N)$ holds for $p=2$.

By the Sobolev embedding theorem, one gets that $L^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ continuously. Further, from $H_V^s \hookrightarrow L^s(\mathbb{R}^N)$ continuously, we deduce that $H_V^s \hookrightarrow L^2(\mathbb{R}^N)$ continuously.

Suppose that $\{u_n\} \subset H_V^s$ is a sequence such that $u_n \rightharpoonup 0$ in H_V^s. Then one gets $u_n \rightharpoonup 0$ in $H^s(\mathbb{R}^N)$ and $u_n \to 0$ in $L^2(B_R)$, where B_R is a ball in $\mathbb{R}^N$ with radius R centered at $0 \in \mathbb{R}^N$.

From condition (*C*), one gets $\lim_{|x|\to\infty} V(x) = \infty$, it follows that for any $\varepsilon > 0$, there exists $R > 0$ such that

$$\left|\frac{1}{V(x)}\right| \le \varepsilon \quad \text{for } |x| > R.$$

Thus,

$$\begin{aligned}\int_{\mathbb{R}^N} |u_n|^2 \,\mathrm{d}x &= \int_{B_R} |u_n|^2 \,\mathrm{d}x + \int_{B_R^c} |u_n|^2 \,\mathrm{d}x \\ &\le \varepsilon + \varepsilon \int_{B_R^c} |V(x)||u_n|^2 \,\mathrm{d}x \\ &\le \varepsilon + C\varepsilon \left(\sup \int_{\mathbb{R}^N} |V(x)||u_n|^2 \,\mathrm{d}x\right).\end{aligned}$$

From this, we conclude that

$$u_n \to 0 \text{ in } L^2(\mathbb{R}^N).$$

Thus, $H_V^s \hookrightarrow L^2(\mathbb{R}^N)$ is compact.

Step 2. We prove the case of $p > 2$.

Since $H_V^s \hookrightarrow L^2(\mathbb{R}^N)$ is compact by **Step 1**, one obtains that $u_n \to 0$ in $L^2(\mathbb{R}^N)$. By the following fractional Gagliardo-Nirenberg-Sobolev inequality ($p \in [2,4)$),

$$\int_{\mathbb{R}^N} |u|^p \,\mathrm{d}x \le \frac{p}{|Q|_2^{p-2}} \alpha_p \beta_p \left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u_n\right|^2 \mathrm{d}x\right)^{\frac{N(p-2)}{4s}} \left(\int_{\mathbb{R}^N} |u_n|^2 \,\mathrm{d}x\right)^{\frac{2ps-N(p-2)+4s}{4s}},$$

we can deduce that $u_n \to 0$ in $L^p(\mathbb{R}^N)$ for $p \in [2,4)$. □

The following lemma is adapted from ([16], Lemma 2.5), for readers' convenience, we provide a brief proof.

Lemma 2.4. *Assume that* $2 < q < p = 4, c > 0, d > 0$ *and the energy functional* $E_0^c(u)$ *is defined as*

$$E_0^c(u) = \frac{a}{2}\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x + \frac{b}{4}\left(\int_{\mathbb{R}^N} \left|(-\Delta)^{\frac{s}{2}} u\right|^2 \mathrm{d}x\right)^2 - \frac{c}{4}\int_{\mathbb{R}^N} |u|^4 \,\mathrm{d}x - \frac{d}{q}\int_{\mathbb{R}^N} |u|^q \,\mathrm{d}x.$$

Set $c_* = bS_s^2$, $m_0(c) = \inf_{u\in S_0} E_0^c(u)$. *Then* $m_0(c) = -\infty$ *for* $c > c_*$, *where*

$$S_0 = \left\{u \in H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 \,\mathrm{d}x = 1\right\}.$$

Proof. Choosing $x_0 = 0$ in $\tilde{u}$. Then

$$U=\frac{\tilde{u}}{|\tilde{u}|_{2_s^*}}=\frac{\mathcal{K}\left(\mu^2+\left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{N-2s}{2}}}{|\tilde{u}|_{2_s^*}}=\theta\left(\mu^2+\left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{N-2s}{2}},$$

where $\theta=\frac{\mathcal{K}}{|\tilde{u}|_{2_s^*}}$. By the assumption of $N=4s$, it yields that for $R>1$,

$$\begin{aligned}\int_{R<|x|<2R}U^2\mathrm{d}x&=\int_{R<|x|<2R}\left[\theta\left(\mu^2+\left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{N-2s}{2}}\right]^2\mathrm{d}x\\&\le S_s^2\theta^2\omega_N\int_{R<r<2R}r^{N-1}r^{-4s}\mathrm{d}r\\&<2S_s^2\theta^2\omega_N:=A_1,\end{aligned}\tag{2.5}$$

where ω_N is the surface area of unit sphere in $\mathbb{R}^N$, and

$$\int_{|x|>R}\left|(-\Delta)^{\frac{s}{2}}U\right|^2\mathrm{d}x\le\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}U\right|^2\mathrm{d}x=S_s^2.\tag{2.6}$$

$$\begin{aligned}\int_{|x|>R}U^4\mathrm{d}x&=\int_{|x|>R}\left[\theta\left(\mu^2+\left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{N-2s}{2}}\right]^4\mathrm{d}x\\&\le\theta^4\omega_N\int_{r>R}r^{N-1}\left(\left|\frac{r}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-4s}\mathrm{d}r\\&=\frac{S_s^4\theta^4\omega_N}{NR^N}:=\frac{A_2}{R^N}.\end{aligned}\tag{2.7}$$

For $2<q<4$, there is

$$\begin{aligned}\int_{|x|>R}U^q\mathrm{d}x&=\int_{|x|>R}\left[\theta\left(\mu^2+\left|\frac{x}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-\frac{N-2s}{2}}\right]^q\mathrm{d}x\\&\le\theta^q\omega_N\int_{r>R}r^{N-1}\left(\left|\frac{r}{S_s^{\frac{1}{2s}}}\right|^2\right)^{-qs}\mathrm{d}r\\&=\frac{S_s^q\theta^q\omega_N}{NR^N}:=\frac{A_3}{R^N}.\end{aligned}\tag{2.8}$$

Since $0<s<1$ and $N=4s$, we consider the computation of $\int_{|x|<R}U^2\mathrm{d}x$ in three cases as below:

1) If $N=1$, then $s=\frac{1}{4}$ and

$$\int_{|x|<R} U^2 \mathrm{d}x = \theta^2 \omega_1 S_s^2 \left(\ln\left| R + \sqrt{R^2 + \left(\mu S_s^2\right)^2} \right| - \ln\left|\mu S_s^2\right| \right).$$

2) If $N=2$, then $s=\frac{1}{2}$ and

$$\int_{|x|<R} U^2 \mathrm{d}x = \frac{\theta^2 \omega_2 S_s^2}{2} \left(\ln\left| R^2 + \left(\mu S_s\right)^2 \right| - \ln\left| \left(\mu S_s\right)^2 \right| \right).$$

3) If $N=3$, then $s=\frac{3}{4}$ and

$$\int_{|x|<R} U^2 \mathrm{d}x = \theta^2 \omega_3 \int_{r<R} r^2 \left(\mu^2 + \left| \frac{r}{S_s^{\frac{2}{3}}} \right|^2 \right)^{-\frac{3}{2}} \mathrm{d}r$$

$$= S_s^2 \theta^2 \omega_3 \left(\ln\left| R + \sqrt{R^2 + \left(S_s^{\frac{2}{3}} \mu \right)^2} \right| - \ln\left| S_s^{\frac{2}{3}} \mu \right| - \frac{R}{\sqrt{R^2 + \left(S_s^{\frac{2}{3}} \mu \right)^2}} \right).$$

So, there exists a $\rho > 0$ such that

$$\int_{|x|<R} U^2 \mathrm{d}x \ge \rho \ln\left(R^2 + \mu S_s^2 \right). \tag{2.9}$$

Then we consider a radially symmetric cut-off function $\phi \in C_0^\infty\left(\mathbb{R}^N\right)$, which satisfies $\phi = 1$ in $B_R = \left\{ x \in \mathbb{R}^N : |x| \le R \right\}$, $\phi = 0$ in $B_{2R}^c = \left\{ x \in \mathbb{R}^N : |x| > 2R \right\}$ and $0 \le \phi \le 1$, $|\nabla \phi| \le \frac{2}{R}$.

Set $\bar{U} = \frac{\phi U}{|\phi U|_2}$, $U_\lambda = \lambda^{\frac{N}{2}} \bar{U}(\lambda x)$. Then it's easy to get that $\bar{U}, U_\lambda \in S_0$ and

$$E_0^c\left(U_\lambda\right) = \frac{a\lambda^{2s}}{2|\phi U|_2^2} \int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} (\phi U) \right|^2 \mathrm{d}x + \frac{b\lambda^{4s}}{4|\phi U|_2^4} \left(\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} (\phi U) \right|^2 \mathrm{d}x \right)^2$$
$$- \frac{c\lambda^N}{4|\phi U|_2^4} \int_{\mathbb{R}^N} |\phi U|^4 \mathrm{d}x - \frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q} \int_{\mathbb{R}^N} |\phi U|^q \mathrm{d}x.$$

According to the definition of function ϕ, it follows that

$$\int_{\mathbb{R}^N} \left| (-\Delta)^{\frac{s}{2}} (\phi U) \right|^2 \mathrm{d}x = \left(\int_{|x|\le R} + \int_{R<|x|\le 2R} \right) \left| (-\Delta)^{\frac{s}{2}} (\phi U) \right|^2 \mathrm{d}x$$
$$= \int_{|x|\le R} \left| (-\Delta)^{\frac{s}{2}} U \right|^2 \mathrm{d}x + \int_{R<|x|\le 2R} \left| (-\Delta)^{\frac{s}{2}} (\phi U) \right|^2 \mathrm{d}x,$$

where

$$\begin{aligned}
&\int_{R<|x|\le 2R}\left|(-\Delta)^{\frac{s}{2}}(\phi U)\right|^2 \mathrm{d}x \\
&=\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{\phi(x)^2|U(x)-U(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\quad+\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{U(y)^2|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\quad+2\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{\phi(x)U(y)(U(x)-U(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y.
\end{aligned}$$

In addition, we get the following estimate

$$\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{\phi(x)^2|U(x)-U(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\le\int_{R<|x|\le 2R}\left|(-\Delta)^{\frac{s}{2}}U\right|^2\mathrm{d}x,$$

$$\begin{aligned}
&\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{U(y)^2|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\le\frac{C}{R^2}\int_{\mathbb{R}^N}\int_{\{x\in\mathbb{R}^N:R<|x|\le 2R,|x-y|\le R\}}\frac{U(y)^2}{|x-y|^{N+2s-2}}\mathrm{d}x\mathrm{d}y \\
&\quad+4\int_{\mathbb{R}^N}\int_{\{x\in\mathbb{R}^N:R<|x|\le 2R,|x-y|\ge R\}}\frac{U(y)^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\le\frac{A_4}{R^{2s}}.
\end{aligned}$$

And it follows from Hölder inequality that

$$\begin{aligned}
&2\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{\phi(x)U(y)(U(x)-U(y))(\phi(x)-\phi(y))}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y \\
&\le C\left(\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{U(y)^2|\phi(x)-\phi(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \\
&\quad\times\left(\int_{\mathbb{R}^N}\int_{R<|x|\le 2R}\frac{\phi(x)^2|U(x)-U(y)|^2}{|x-y|^{N+2s}}\mathrm{d}x\mathrm{d}y\right)^{\frac{1}{2}} \\
&\le\frac{A_5}{R^s}.
\end{aligned}$$

Choosing $L=\max\{A_4,A_5\}$, we obtain that

$$\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}(\phi U)\right|^2\mathrm{d}x\le\int_{|x|<2R}\left|(-\Delta)^{\frac{s}{2}}U\right|^2\mathrm{d}x+\frac{2L}{R^s}.$$

In addition,

$$\int_{\mathbb{R}^N}|U\phi|^q\mathrm{d}x\ge\int_{|x|\le R}|U|^q\mathrm{d}x\ge\int_{\mathbb{R}^N}|U|^q\mathrm{d}x-\frac{A_3}{R^N},$$

where $\int_{\mathbb{R}^N}|U|^q\mathrm{d}x$ is bounded from above by (2.8) and Hölder inequality. Then one gets

$$E_0^c(U_\lambda)\le\frac{a\lambda^{2s}}{2|\phi U|_2^2}\left(S_s^2+\frac{2L}{R^s}\right)+\frac{\lambda^{4s}}{4|\phi U|_2^4}\left(\left(bS_s^2-c\right)S_s^2+\frac{4bLS_s^2}{R^s}+\frac{4bL^2}{R^{2s}}+\frac{cA_2}{R^N}\right)$$
$$-\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\int_{\mathbb{R}^N}|U|^q\,\mathrm{d}x+\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\frac{A_3}{R^N}. \quad (2.10)$$

When $c>c_*=bS_s^2$ and R is sufficiently large, we get

$$\frac{4bLS_s^2}{R^s}+\frac{4bL^2}{R^{2s}}+\frac{cA_2}{R^N}<\frac{1}{2}\left(c-bS_s^2\right)S_s^2.$$

Thus, it follows from (2.10) that $m_0(c)\le E_0^c(u)\to-\infty$ as $\lambda\to\infty$, *i.e.*, $m_0(c)=-\infty$ for $c>c_*$. □

In the following lemma, we give the estimates of c_* for $p=4$.

Lemma 2.5. *Suppose* $2<q<p=4$, *d is a positive constant and* $V(x)$ *satisfies condition* (*C*). *Let* $c_*=bS_s^2$, *then*

$$\begin{cases} m_V(c)>0, & \text{if } c\in(0,c_*),\\ m_V(c)=0, & \text{if } c=c_*,\\ m_V(c)=-\infty, & \text{if } c\in(c_*,\infty).\end{cases}$$

Proof. 1) If $c<c_*=bS_s^2$, we set $\gamma=\min\left\{\varepsilon_0,bS_s^2-c-\varepsilon_0\right\}$ and choose $\varepsilon_0<bS_s^2-c$ satisfying condition (*C*) in Theorem 1.1. Firstly, by Hölder inequality and $u\in S_V$, we have

$$\begin{aligned}\int_{\mathbb{R}^N}|u|^q\,\mathrm{d}x&=\int_{\mathbb{R}^N}|u|^{2(q-2)}|u|^{q-2(q-2)}\,\mathrm{d}x\\&\le\left(\int_{\mathbb{R}^N}\left(|u|^{2(q-2)}\right)^{\frac{2}{q-2}}\mathrm{d}x\right)^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^N}\left(|u|^{q-2(q-2)}\right)^{\frac{2}{4-q}}\mathrm{d}x\right)^{\frac{4-q}{2}}\\&\le\left(\int_{\mathbb{R}^N}|u|^4\,\mathrm{d}x\right)^{\frac{q-2}{2}}\left(\int_{\mathbb{R}^N}|u|^2\,\mathrm{d}x\right)^{\frac{4-q}{2}}\\&=\left(\int_{\mathbb{R}^N}|u|^4\,\mathrm{d}x\right)^{\frac{q-2}{2}}.\end{aligned}$$

Then by Sobolev inequality, Young's inequality, Hölder inequality and condition (*C*), we get for any $u\in S_V$,

$$\begin{aligned}E_V^c(u)&=\frac{a}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2+\frac{b}{4}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^4+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{c}{4}|u|_4^4-\frac{d}{q}|u|_q^q\\&\ge\frac{b}{4}S_s^2|u|_4^4-\frac{c}{4}|u|_4^4+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\&\ge\frac{\gamma}{2}-\frac{1}{2}\int_{V(x)\le\gamma}(\gamma-V(x))u^2\mathrm{d}x+\frac{b}{4}S_s^2|u|_4^4-\frac{c}{4}|u|_4^4-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\&=\frac{\gamma}{2}-\frac{1}{2}\int_{V(x)\le\gamma}\left[\frac{1}{\gamma}(\gamma-V(x))^2\right]^{\frac{1}{2}}\left[\gamma u^4\right]^{\frac{1}{2}}\mathrm{d}x+\frac{1}{4}\left(bS_s^2-c\right)|u|_4^4-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\&\ge\frac{\gamma}{2}-\frac{1}{2}\int_{V(x)\le\gamma}\left[\frac{\frac{1}{\gamma}(\gamma-V(x))^2}{2}+\frac{\gamma u^4}{2}\right]\mathrm{d}x+\frac{1}{4}\left(bS_s^2-c\right)|u|_4^4-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\&=\frac{\gamma}{2}-\frac{1}{4\gamma}\int_{V(x)\le\gamma}(\gamma-V(x))^2\mathrm{d}x+\frac{bS_s^2-c-\gamma}{4}|u|_4^4-d\left(|u|_4^4\right)^{\frac{q-2}{2}}.\end{aligned}$$

Let $f(t)=\frac{bS_s^2-c-\gamma}{4}t-dt^{\frac{q-2}{2}}$, by direct computation, one gets

$$f'(t)=\frac{bS_s^2-c-\gamma}{4}-\frac{(q-2)d}{2}t^{\frac{q-4}{2}}.$$

When $f'(t)=0$, there is

$$t=\left(\frac{2(q-2)d}{bS_s^2-c-\gamma}\right)^{\frac{2}{4-q}}$$

and

$$\min_{t\geq 0} f(t)=d\left(\frac{q-2}{2}-1\right)\left(\frac{2(q-2)d}{bS_s^2-c-\gamma}\right)^{\frac{q-2}{4-q}}.$$

Thus, we get

$$\begin{aligned}E_V^c(u)&\geq\frac{\gamma}{2}-\frac{\gamma}{4}\varepsilon_0+d\left(\frac{q-2}{2}-1\right)\left(\frac{2(q-2)d}{bS_s^2-c-\gamma}\right)^{\frac{q-2}{4-q}}\\&=\frac{\gamma}{4}(2-\varepsilon_0)+d\left(\frac{q-2}{2}-1\right)\left(\frac{2(q-2)d}{bS_s^2-c-\gamma}\right)^{\frac{q-2}{4-q}}\\&\geq\frac{\gamma}{4}+d\left(\frac{q-2}{2}-1\right)\left(\frac{2(q-2)d}{bS_s^2-c-\gamma}\right)^{\frac{q-2}{4-q}}\\&=\gamma\left[\frac{1}{4}+d\left(\frac{q-2}{2}-1\right)\left[2d(q-2)\right]^{\frac{q-2}{4-q}}\frac{1}{\gamma}\left(\frac{1}{bS_s^2-c-\gamma}\right)^{\frac{q-2}{4-q}}\right].\end{aligned}$$

If $\gamma=\varepsilon_0$, namely $\varepsilon_0<bS_s^2-c-\varepsilon_0$, we set $d>0$ satisties the condition that

$$d\left(1-\frac{q-2}{2}\right)\left[2d(q-2)\right]^{\frac{q-2}{4-q}}\frac{1}{\varepsilon_0}\left(\frac{1}{bS_s^2-c-\varepsilon_0}\right)^{\frac{q-2}{4-q}}<\frac{1}{8};$$

and if $\gamma=bS_s^2-c-\varepsilon_0$, *i.e.*, $bS_s^2-c-\varepsilon_0<\varepsilon_0$, we set $d>0$ satisties the condition that

$$d\left(1-\frac{q-2}{2}\right)\left[2d(q-2)\right]^{\frac{q-2}{4-q}}\frac{1}{bS_s^2-c-\varepsilon_0}\left(\frac{1}{\varepsilon_0}\right)^{\frac{q-2}{4-q}}<\frac{1}{8}.$$

Then we get $E_V^c(u)\leq\frac{\gamma}{8}$. This indicates that $m_V(c)>0$ for all $c<c_*=bS_s^2$.

2) If $c>c_*=bS_s^2$, then by (2.10) of Lemma 2.4, we obtain

$$\begin{aligned}E_V^c(U_\lambda)&=E_0^c(U_\lambda)+\frac{1}{2}\int_{\mathbb{R}^N}V(x)U_\lambda^2\mathrm{d}x\\&\leq\frac{a\lambda^{2s}}{2|\phi U|_2^2}\left(S_s^2+\frac{2L}{R^s}\right)-\frac{\lambda^{4s}}{4|\phi U|_2^4}\Bigg(\left(c-bS_s^2\right)S_s^2-\frac{4bLS_s^2}{R^s}\\&\quad-\frac{4bL^2}{R^{2s}}-\frac{cA_2}{R^N}-2|\phi U|_2^2\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\mathrm{d}x\Bigg)\\&\quad-\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\int_{\mathbb{R}^N}|U|^q\mathrm{d}x+\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\frac{A_3}{R^N}.\end{aligned}$$

Similar to the proof of Lemma 2.4, we choose R large enough such that

$$\frac{4bLS_s^2}{R^s}+\frac{4bL^2}{R^{2s}}+\frac{cA_2}{R^N}<\frac{1}{2}\left(c-bS_s^2\right)S_s^2.$$

For $\lambda>1$, we obtain that

$$\phi(\lambda x)U(\lambda x)\le\phi(x)U(x),\ \phi(\lambda x)U(\lambda x)\to 0 \text{ as } \lambda\to\infty.$$

It follows from the Lebesgue dominated convergence theorem that

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\,\mathrm{d}x=0.$$

Therefore, according to the above inequalities and the definition of infimum, we deduce that

$$m_V(c)\le\lim_{\lambda\to\infty}E_V^c(U_\lambda)=-\infty.$$

3) If $c=c_*=bS_s^2$, then using the Sobolev inequality and Hölder inequality as in case (1), we have

$$\begin{aligned}E_V^c(u)&=\frac{a}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2+\frac{b}{4}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^4-\frac{c}{4}|u|_4^4+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{d}{q}|u|_q^q\\&\ge\frac{a}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x+\frac{1}{4}\left(bS_s^2-c\right)|u|_4^4-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\&=\frac{a}{2}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2-dS_s^{2-q}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2(q-2)}+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x.\end{aligned}$$

Choosing appropriate $a>0$ (Actually, we set

$$a\ge\frac{2dS_s^{2-q}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2(q-2)}-\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x}{\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2}$$

and $a>0$ in Theorem), we conclude $m_V(c)\ge 0$ for $c=c_*$.

Further, we prove that $m_V(c)\le 0$ for $c=c_*$. Analogous to the proof of Lemma 2.4, we get that there exists a $R(\varepsilon)>0$ satisfying for $R\ge R(\varepsilon)$,

$$|\phi U|_2^2\ge\int_{|x|<R}|U|^2\,\mathrm{d}x\ge\rho\ln\left(R^2+\mu S_s^2\right)\ge\frac{1}{\varepsilon},\left|(-\Delta)^{\frac{s}{2}}U\right|_2^2=S_s^2\ge\frac{2L}{R^s}.$$

Repeating the previous proof, we get

$$\lim_{\lambda\to\infty}\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\,\mathrm{d}x=0.$$

So, there exists $\lambda(\varepsilon)>0$ such that

$$\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\,\mathrm{d}x\le\varepsilon^2\quad\text{for }\lambda\ge\lambda(\varepsilon).$$

According to the previous analysis, we obtain

$$\begin{aligned}m_V(c_*)&\le E_V^{c_*}(U_\lambda)\\&\le\frac{a\lambda^{2s}}{2|\phi U|_2^2}\left(S_s^2+\frac{2L}{R^s}\right)+\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\frac{A_3}{R^N}-\frac{d\lambda^{2s(q-2)}}{q|\phi U|_2^q}\int_{\mathbb{R}^N}|U|^q\,\mathrm{d}x\end{aligned}$$

$$
\begin{aligned}
&+\frac{\lambda^{4s}}{4|\phi U|_2^4}\left(\frac{4bLS_s^2}{R^s}+\frac{4bL^2}{R^{2s}}+\frac{cA_2}{R^N}+2|\phi U|_2^2\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\,\mathrm{d}x\right)\\
&\le C_1\varepsilon\lambda^{2s}+\frac{\lambda^{4s}}{4|\phi U|_2^4}\left(\frac{4bLS_s^2}{R^s}+\frac{4bL^2}{R^{2s}}+\frac{cA_2}{R^N}\right)\\
&+\frac{\lambda^{4s}}{2|\phi U|_2^2}\int_{\mathbb{R}^N}V(x)\phi(\lambda x)^2U(\lambda x)^2\,\mathrm{d}x+C_2\varepsilon^2\lambda^{2s(q-2)}-C_3\varepsilon\lambda^{2s(q-2)}\\
&\le\tilde{C}\varepsilon\lambda^{2s}+C'\varepsilon^3\lambda^{4s}+C_2'\varepsilon^2\lambda^{2s(q-2)}-C_3'\varepsilon\lambda^{2s(q-2)},
\end{aligned}
$$

where $C_1, C_2, C_3, \tilde{C}, C', C_2', C_3'$ are all positive constants.

Setting $\lambda=\max\left\{\lambda(\varepsilon),\varepsilon^{-\frac{1}{4s}}\right\}$, we get $m_V(c_*)\le C\varepsilon^{\frac{1}{2}}$, namely, $m_V(c_*)\le 0$. Thus, we deduce $m_V(c_*)=0$. □

3. Proof of the Main Result

In this section, we prove the main result of this paper.

Proof of Theorem 1.1. By Lemma 2.5, we only need to prove that $E_V^c(u)$ has no minimizers for $c=c_*$ and it has an energy minimizer for all $c<c_*$ when $2<q<p=4$.

If $p=4$ and $c=c_*$, using the fractional Sobolev inequality (2.3) and the Hölder inequality as in the proof of Lemma 2.5 and the assumption about a in Theorem 1.1, we deduce

$$
\begin{aligned}
E_V^{c_*}(u)&\ge\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{d}{q}|u|_q^q\\
&+\frac{b}{4}\left(1-\frac{c_*}{c_*}\right)\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x\right)^2\\
&=\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-\frac{d}{q}|u|_q^q\\
&\ge\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-d\left(|u|_4^4\right)^{\frac{q-2}{2}}\\
&\ge\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-dS_s^{2-q}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2(q-2)}\ge 0.
\end{aligned}
$$

Arguing by contradiction that m_V^c can be obtained for $c=c_*$. Then, by Lemma 2.5, we deduce that

$$
0=m_V(c_*)=E_V^{c_*}(u)\ge\frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-dS_s^{2-q}\left|(-\Delta)^{\frac{s}{2}}u\right|_2^{2(q-2)}.
$$

From this, we get $u=0$ in $D^{s,2}(\mathbb{R}^N)$. So $u=0$ in $L^2(\mathbb{R}^N)$, which is in contradiction with $|u|_2=1$. Therefore, $E_V^c(u)$ has no minimizers for $c=c_*$.

If $p=4$ and $c<c_*$, let $\{u_n\}\subset S_V$ be a minimizing sequence for $m_V(c)$, then by the Fractional Sobolev inequality (2.3), we deduce that

$$
\begin{aligned}
E_V^c(u_n) &= \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 \mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x \\
&\quad +\frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 \mathrm{d}x\right)^2-\frac{c}{4}|u_n|_4^4-\frac{d}{q}|u_n|_q^q \\
&\ge \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 \mathrm{d}x+\frac{1}{2}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x \\
&\quad +\frac{bS_s^2-c}{4}\int_{\mathbb{R}^N}|u_n|^4\mathrm{d}x-\frac{d}{q}\int_{\mathbb{R}^N}|u_n|^q\mathrm{d}x.
\end{aligned}
$$

From this, we can deduce $\{u_n\}$ is bounded in H_V^s. According to Lemma 2.3, we may assume that there exists a $u\in H_V^s$ such that $u_n \rightharpoonup u$ in H_V^s and $u_n\to u$ in $L^p(\mathbb{R}^N)$ for $p\in(2,4)$, then $u\in S_V$ and there exists $\xi_n\in\mathbb{R}$ such that

$$
\lim_{n\to\infty}\left(E_V^{c\prime}(u_n)-\xi_n u_n\right)=0.
$$

Set $\lim_{n\to\infty}\left|(-\Delta)^{\frac{s}{2}}u_n\right|_2^2=B$. Then

$$
\begin{aligned}
\lim_{n\to\infty}\xi_n &= \lim_{n\to\infty}\left(E_V^{c\prime}(u_n),u_n\right)=\lim_{n\to\infty}\left[\left(E_V^{c\prime}(u_n),u_n\right)-4E_V^c(u_n)+4m_V(c)\right] \\
&= \lim_{n\to\infty}\left(4m_V(c)-a\left|(-\Delta)^{\frac{s}{2}}u_n\right|_2^2-\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x-\frac{d(q-4)}{q}\int_{\mathbb{R}^N}|u_n|^q\mathrm{d}x\right) \\
&:= \xi.
\end{aligned}
$$

Choosing $\phi\in H_V$, we get

$$
\begin{aligned}
0 &= \lim_{n\to\infty}\left(E_V^{c\prime}(u_n)-\xi_n u_n,\phi\right) \\
&= \lim_{n\to\infty}\left(\left(E_V^{c\prime}(u_n),\phi\right)-\xi_n\int_{\mathbb{R}^N}u_n\phi\mathrm{d}x\right) \\
&= \lim_{n\to\infty}\left(\left(a+b\left|(-\Delta)^{\frac{s}{2}}u_n\right|_2^2\right)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u_n(-\Delta)^{\frac{s}{2}}\phi\mathrm{d}x+\int_{\mathbb{R}^N}V(x)u_n\phi\mathrm{d}x\right. \\
&\quad \left.-c\int_{\mathbb{R}^N}u_n^3\phi\mathrm{d}x-d\int_{\mathbb{R}^N}u_n^{q-1}\phi\mathrm{d}x-\xi_n\int_{\mathbb{R}^N}u_n\phi\mathrm{d}x\right) \\
&= (a+bB)\int_{\mathbb{R}^N}(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}\phi\mathrm{d}x+\int_{\mathbb{R}^N}V(x)u\phi\mathrm{d}x-c\int_{\mathbb{R}^N}u^3\phi\mathrm{d}x \\
&\quad -d\int_{\mathbb{R}^N}u^{q-1}\phi\mathrm{d}x-\xi\int_{\mathbb{R}^N}u\phi\mathrm{d}x.
\end{aligned} \tag{3.1}
$$

If we choose $\phi=u$ in (3.1), then

$$
\xi=(a+bB)\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2\mathrm{d}x+\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x-c\int_{\mathbb{R}^N}u^4\mathrm{d}x-d\int_{\mathbb{R}^N}u^q\mathrm{d}x.
$$

Since $u\in S_V$ and $\left|(-\Delta)^{\frac{s}{2}}u\right|_2^2\le\liminf_{n\to\infty}\left|(-\Delta)^{\frac{s}{2}}u_n\right|_2^2=B$, we obtain

$$
\begin{aligned}
m_V(c) \le E_V^c(u) &= \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x \\
&\quad + \frac{b}{4}\left(\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x\right)^2 - \frac{c}{4}|u|_4^4 - \frac{d}{4}\int_{\mathbb{R}^N}u^q\mathrm{d}x - \frac{d(4-q)}{4q}\int_{\mathbb{R}^N}u^q\mathrm{d}x \\
&\le \frac{a}{2}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x + \frac{1}{2}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x + \frac{bB}{4}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x \\
&\quad - \frac{c}{4}|u|_4^4 - \frac{d}{4}\int_{\mathbb{R}^N}u^q\mathrm{d}x - \frac{d(4-q)}{4q}\int_{\mathbb{R}^N}u^q\mathrm{d}x \\
&\le \frac{a}{4}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u\right|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}V(x)u^2\mathrm{d}x + \frac{1}{4}\xi - \frac{d(4-q)}{4q}|u|_q^q \\
&\le \liminf_{n\to\infty}\left(\frac{a}{4}\int_{\mathbb{R}^N}\left|(-\Delta)^{\frac{s}{2}}u_n\right|^2 \mathrm{d}x + \frac{1}{4}\int_{\mathbb{R}^N}V(x)u_n^2\mathrm{d}x + \frac{1}{4}\xi_n + \frac{d(q-4)}{4q}|u_n|_q^q\right) \\
&= m_V(c).
\end{aligned}
$$

Therefore, $E_V^c(u) = m_V(c)$ and $E_V^c(u)$ has an energy minimizer *u*.

4. Conclusion

In summary, in the previous sections, combining the fractional Gagliardo-Nirenberg-Sobolev inequality, and fractional Sobolev inequality with some energy estimates, we obtained the existing result of the fractional Kirchhoff equation with doubly critical exponents and mixed nonlinearities. □

Conflicts of Interest

The author declares no conflicts of interest.

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