Applied Mathematics, 2021, 12, 500-519
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Quasilinear Degenerated Elliptic Systems with Weighted in Divergence Form with Weak Monotonicity with General Data

Abdelkrim Barbara, El Houcine Rami, Elhoussine Azroul

Laboratory LAMA, Department of Mathematics, Faculty of Sciences Dhar El Mahraz, Sidi Mohammed Ben Abdellah University, Atlas Fez, Morocco
Email: babdelkarim66@hotmail.com, ramielhoucine@gmail.com, elhoussine.azroul@gmail.com

How to cite this paper: Barbara, A., Rami, E.H. and Azroul, E. (2021) Quasilinear Degenerated Elliptic Systems with Weighted in Divergence Form with Weak Monotonicity with General Data. *Applied Mathematics*, **12**, 500-519.
https://doi.org/10.4236/am.2021.126035

Received: October 11, 2020
Accepted: June 27, 2021
Published: June 30, 2021

Copyright © 2021 by author(s) and Scientific Research Publishing Inc.
This work is licensed under the Creative Commons Attribution International License (CC BY 4.0).
http://creativecommons.org/licenses/by/4.0/
Open Access

Abstract

We consider, for a bounded open domain Ω in $I\!R^n$ and a function $u:\Omega \to I\!R^m$, the quasilinear elliptic system:

$$(QES)_{(f,g)} \begin{cases} -div\sigma(x,u(x),Du(x)) = v(x) + f(x,u,Du) + divg(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega, \end{cases}$$

(1). We generalize the system $(QES)_{(f,g)}$ in considering a right hand side depending on the jacobian matrix Du. Here, the star in $(QES)_{(f,g)}$ indicates that f may depend on Du. In the right hand side, v belongs to the dual space $W^{-1,p'}(\Omega,\omega^*,R^m)$, $\left(\frac{1}{p}+\frac{1}{p'}=1,\ p>1\right)$, f and g satisfy some standard continuity and growth conditions. We prove existence of a regularity, growth and coercivity conditions for σ, but with only very mild monotonicity assumptions.

Keywords

Quasilinear Elliptic, Sobolev Spaces with Weight, Young Measure, Galerkin Scheme

1. Introduction

In this paper, the main point is that we do not require monotonicity in the strict monotonicity of a typical Leray-Lions operator as it is usually assumed in previous papers. The aims of this text are to prove analogous existence results under relaxed monotonicity, in particular under strict quasi-monotonicity. The main technical tool we advocate and use throughout the proof is Young measures. By applying a Galerkin schema, we obtain easily an approximating sequence u_k. The Ball theorem [1] and especially the resulting tool mode available by Hungerbühler to partial differential equation theory give them a sufficient control on the gradient approximating sequence Du_k to pass to the limit. This method is used by Dolzmann [2], G. J. Minty [3], H. Brezis [4], H. E. Stromberg [5], Muller [6], J. L. Lions [7], Kristznsen, J. Lower [8], M. I. Visik [9] and mainly by Hungurbühler to get the existence of a weak solution for the quasi-linear elliptic system [10]. This paper can be seen as generalization of Hungurbühler and as a continuation of Y-Akdim [11].

This kind of problems finds its applications in the model of Thomas-Fermis in atomic physics [12], and also porous flow modeling in reservoir [13].

2. Priliminaries

Let $\omega=\{\omega_{ij};0\le i\le n;1\le j\le m\}$ the weight function systems defined in Ω and satisfying the following integrability conditions:

$$\begin{aligned}&\omega_{ij}\in L^1_{loc}(\Omega),\ \omega_{ij}^{\frac{-1}{p-1}}\in L^1_{loc}(\Omega),\ \text{for some } p\in\,]1,\infty[\\&\text{and } \exists s>0 \text{ such that } \omega_{ij}^{-s}\in L^1(\Omega).\end{aligned}\tag{2.1}$$

with $\omega^*=\{\omega^*_{ij}=\omega_{ij}^{1-p'},0\le i\le n,1\le j\le m\}$, $\sigma=(\sigma_{rs})$ with $1\le s\le n, 1\le r\le m$ and which satisfies some hypotheses (see below).

We denote by $I\!M^{m\times n}$ the real vector space of $m\times n$ matrices equipped with the inner product $M:N=\sum_{ij}M_{ij}N_{ij}$.

The Jacobian matrix of a function $u:\Omega\to I\!R^m$ is denoted by $Du(x)=(D_1u(x),D_2u(x),\cdots,D_nu(x))$ with $D_i=\partial/\partial(x_i)$.

The space $W^{1,p}(\Omega,\omega,I\!R^m)$ is the set of functions

$$\{u=u(x)/u\in L^p(\Omega,\overline{\omega_0},I\!R^m)\},\ D_{ij}u=\frac{\partial u^i}{\partial x_j}\in L^p(\Omega,\omega_{ij},I\!R^m),$$
$$1\le i\le n,1\le j\le m.$$

with

$$L^p(\Omega,\omega_{ij},I\!R^m)=\left\{u=u(x)/|u|\,\omega_{ij}^{\frac{1}{p}}\in L^p(\Omega,I\!R^m)\right\}$$

The weighted space $W^{1,p}(\Omega,\omega,I\!R^m)$ can be equipped by the norm:

$$\|u\|_{1,p,\omega}=\left(\sum_{j=1}^m\int_\Omega|u_j|^p\,\omega_{0j}\mathrm{d}x+\sum_{1\le i\le n,1\le j\le m}\int_\Omega|D_{ij}u|^p\,\omega_{ij}\mathrm{d}x\right)^{\frac{1}{p}}$$

where $\overline{\omega_0}=(\omega_{0j})$ and $1\le j\le m$. the norm $\|\cdot\|_{1,\omega,p}$ is equivalent to the norm $|||\cdot|||$, on $W_0^{1,p}(\Omega,\omega,I\!R^m)$, such that, $|||u|||=\left(\sum_{1\le i\le n,1\le j\le m}\int_\Omega|D_{ij}u|^p\,\omega_{ij}\mathrm{d}x\right)^{\frac{1}{p}}$.

Proposition 2.1 *The weighted Sobolev space* $W^{1,p}(\Omega,\omega,I\!R^m)$ *is a Banach space, separable and reflexive. The weighted Sobolev space* $W_0^{1,p}(\Omega,\omega,I\!R^m)$ *is the* *closure of* $C_0^\infty(\Omega,\omega,\mathbb{R}^m)$ *in* $W^{1,p}(\Omega,\omega,\mathbb{R}^m)$ *equipped by the norm* $\|\cdot\|_{1,p,\omega}$.

Proof: The prove of proposition is a slight modification of the analogous one in [14] [Kufner-Drabek].

Definition 2.1 *A Young measure* $(\vartheta_x)_{x\in\Omega}$ *is called* $W^{1,p}$*-gradient young measures* ($1\le p<\infty$) *if it is associated to a sequence of gradients* Du_k *such that* u_k *is bounded in* $W^{1,p}(\Omega)$. *The* $W^{1,p}$*-gradient young measures* $(\vartheta_x)_{x\in\Omega}$ *is called homogeneous, if it doesn't depend on x, i-e, if* $\vartheta_x=\vartheta$ *for a.e.* $x\in\Omega$.

Theorem 2.1 (*Kinder, Lehirer-Pedregal*) *let* $(\upsilon_x)_{x\in\Omega}$, *be a family of probability measures in* $\left(C\left(M^{m\times n}\right)\right)$. *Then,* $(\upsilon_x)_{x\in\Omega}$ *is* $W^{1,p}$ *Young measures if and only if:*

1) *There is a* $u\in W^{1,p}(\Omega,\mathbb{R}^m)$ *such that* $Du(x)=\int_{M^{m\times n}}A\,d\vartheta_x(A)$, *a.e in* Ω.

2) *Jensen's inequality:* $\phi(Du(x))\le\int_{M^{m\times n}}\phi(A)\,d\vartheta_x(A)$ *hold for all* $\phi\in X^p$ *quasi-convex, and.*

3) *The function:* $\psi(x)=\int_{M^{m\times n}}|A|^p\,d\vartheta_x(A)\in L^1(\Omega)$. *Here,* X^p *denotes the (not separable) space:*

$$X^p=\left\{\psi\in C\left(M^{m\times n}\right):|\psi(A)|\le c\times\left(1+|A|^p\right),\text{ for all }A\in M^{m\times n}\right\}.$$

proof: see [15].

Theorem 2.2 (*Ball*) *Let* $\Omega\in\mathbb{R}^n$ *be Lebesgue measurable, let* $K\in\mathbb{R}^m$ *be closed, and let* $u_j:\Omega\to\mathbb{R}^m, j\in\mathbb{N}$, *be a sequence of Lebesgue measurable functions satisfying* $u_j\to K$, *as* $j\to\infty$, *i.e. given any open neighborhood U of* $K\in\mathbb{R}^m$, $\lim_{j\to\infty}\left|x\in\Omega:u_j(x)\in U\right|=0$. *Then there exists a subsequence* u_k *of* u_j *and a family* $\vartheta_x, x\in\Omega$, *of positive measures on* $\mathbb{R}^m$, *depending measurably on x, such that*

1) $\|\vartheta_x\|_M=\int_{\mathbb{R}^m}d\vartheta_x\le 1$, *for a.e* $x\in\Omega$.

2) $supp\,\vartheta_x\subset K$ *for a.e* $x\in\Omega$.

3) $f(u_k)\rightharpoonup^*\langle\vartheta_x,f\rangle=\int_{\mathbb{R}^m}f(\lambda)\,d\vartheta_x(\lambda)$ *in* $L^\infty(\Omega)$. *for each continuous functions* $f:\mathbb{R}^m\to\mathbb{R}$ *satisfying*

$\lim f(\lambda)=0, |\lambda|\to\infty$ [1].

Theorem 2.3 (*vitali*) *Let* $\Omega\in\mathbb{R}^n$ *be an open bounded domain and let* u_n *be a sequence in* $L^p(\Omega,\mathbb{R}^m)$ *with* $1\le p<\infty$.

Then u_n *is a cauchy sequence in the* L^p*-norm if and only if the two following conditions hold:*

1) u_n *is cauchy in measure (i.e.:* $\forall\varepsilon>0$, $\left|\left\{x\in\Omega,|u_n(x)-u_m(x)|\ge\varepsilon\right\}\right|=0$ *as* $m,n\to\infty$.

2) $|u_n|^p$ *is equiintegrable i.e.:*

($sup_n\int_\Omega|u_n|^p\,dx<\infty$ *and* $\forall\varepsilon>0,\exists\delta>0$ *such that* $\int_E|u_n|^p\,dx<\varepsilon$ *for all n whenever* $E\subset\Omega$ *and* $|E|<\delta$). *Note that if* u_n *converges pointiest, then* u_n *is cauchy in measure.*

Hypotheses (H_0) (Hardy-Type inequalities): There exist some constant $c>0$, some weighted function γ and some real q ($1<q<\infty$) such that,

$$\left(\sum_{j=1}^m\int_\Omega|u_j(x)|^q\gamma_j(x)\,dx\right)^{\frac{1}{q}}\le c\left(\sum_{1\le i\le n,1\le j\le m}\int_\Omega|D_{ij}u|^p\omega_{ij}\right)^{\frac{1}{p}},$$

for all $u \in W_0^{1,p}(\Omega,\omega,I\!R^m)$, with $\gamma = \{\gamma_j / 1 \le j \le m\}$.

The injection $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^q(\Omega,\gamma,I\!R^m)$ is compact, and $W_0^{1,p}(\Omega,\omega,I\!R^m) \hookrightarrow\hookrightarrow L^r(\Omega,I\!R^m)$ is compact, (by [14]) with

$$\begin{cases} 1 \le r \prec \dfrac{nps}{n(s+1)-ps} & \text{if } ps \prec n(s+1) \\ r \ge 1 & \text{if } n(s+1) \prec ps \end{cases}$$

(H_1) Continuity: $\sigma : \Omega \times I\!R^m \times I\!M^{m\times n} \to I\!M^{m\times n}$ is a Carathéodory function (i-e $x \mapsto \sigma(x,u,F)$ is measurable for every $(u,F) \in I\!R^m \times I\!M^{m\times n}$ and $(u,F) \mapsto \sigma(x,u,F)$ is continuous for almost every $x \in \Omega$). (H_2) Growths and coercivity conditions: There exist $c_1 \ge 0$, $c_2 > 0$, $\lambda_1 \in L^{p'}(\Omega)$, $\lambda_2 \in L^1(\Omega)$, $\lambda_3 \in L^{(p/\alpha)'}(\Omega)$, $0 < \alpha < p$, $1 < q < \infty$ and $\beta > 0$ such that for all $1 \le r \le n$, $1 \le s \le m$, we have:

$$\left|\sigma_{rs}(x,u,F)\right| \le \beta w_{rs}^{1/p}\left[\lambda_1(x) + c_1\sum_{j=1}^{m}\left|\gamma_j\right|^{1/p'} \cdot \left|u_j\right|^{q/p'} + c_1 \sum_{1\le i\le N, 1\le j\le m} \omega_{ij}^{1/p'}\left|F_{ij}\right|^{p-1}\right] \quad (2.2)$$

and

$$\sigma(x,u,F):F \ge -\lambda_2(x) - \sum_{j=1}^{m}\omega_{0j}(x)^{\alpha/p}\lambda_3(x)\left|u_j\right|^{\alpha} + c_2\sum_{1\le i\le n, 1\le j\le m}\omega_{ij}(x)\cdot\left|F_{ij}\right|^{p} \quad (2.3)$$

(H_3) Monotonicity conditions: σ satisfies one of the following conditions:

1) For all $x \in \Omega$, and all $u \in I\!R^m$, the map $F \mapsto \sigma(x,u,F)$ is a C^1-function and is monotone (i-e, $(\sigma(x,u,F) - \sigma(x,u,G)):(F-G) \ge 0$, for all $x \in \Omega$, all $u \in I\!R^m$ and all $F,G \in I\!M^{m\times n}$).

2) There exists a function $W : \Omega \times I\!R^m \times I\!M^{m\times n} \to I\!M^{m\times n}$ such that $\sigma(x,u,F) = \dfrac{\partial W}{\partial F}(x,u,F)$ and $F \mapsto W(x,u,F)$ is convex and C^1 function.

3) For all $x \in \Omega$, and for all $u \in I\!R^m$ the map $F \mapsto \sigma(x,u,F)$ is strictly monotone (*i.e.*, $\sigma(x,u,.)$ is monotone and: $\left[(\sigma(x,u,F) - \sigma(x,u,G)):(F-G) = 0\right] \Rightarrow F = G$).

4) $\sigma(x,u,F)$ is strictly p-quasi-monotone in *F*, *i.e.*,

$$\int_{I\!M^{m\times n}} \left(\sigma(x,u,\lambda) - \sigma(x,u,\bar{\lambda})\right):(\lambda - \bar{\lambda})\,\mathrm{d}\vartheta(\lambda) > 0,$$

for all homogeneous $W^{1,p,w}$-gradient young measures ϑ with center of mass $\bar{\lambda} = \langle \vartheta, id \rangle$ which are not a single Dirac mass.

The main point is that we do not require strict monotonicity or monotonicity in the variables (u,F) in (H_3) as it is usually assumed in previous work (see [15] or [16]).

$(F_0)^*$: (continuity) $f : \Omega \times I\!R^m \times I\!M^{m\times n} \to I\!R^m$ is a Carathéodory function i-e: $x \mapsto f(x,u,F)$ is measurable for every $u \in I\!R^m$, and $F \in I\!M^{m\times n}$, $(u,F) \mapsto f(x,u,F)$ is continuous for almost every $x \in \Omega$.

$(F_1)^*$: (growth condition): The exist: $b_1 \in L^{p'}(\Omega)$, $c_1' > 0$, $c_2' > 0$ such that:

$$\left|f_j(x,u)\right| \le \left[b_1(x) + c_1'\gamma_j^{\frac{1}{p'}}\left|u_j\right|^{\frac{q}{p'}} + c_2'\sum_{r,s}\omega_{rs}^{\frac{1}{P'}}\left|F_{rs}\right|^{p-1}\right]\omega_{0j}^{\frac{1}{P}};$$

$\forall 1\le j\le m$ $(G_0)^*$: (continuity) the map $g:\Omega\times\mathbb{R}^m\to\mathbb{M}^{m\times n}$ is a Carathéodory function. $(G_1)^*$: (growth condition) There exist: $b_2\in L^{p'}(\Omega)$

$$|g_{rs}|\le\omega_{rs}^{\frac{1}{p}}\left[b_2+\sum_j\gamma_j^{\frac{1}{p'}}|u_j|^{\frac{q}{p'}}\right]$$

For all $1\le r\le n$ and $1\le s\le m$.

Our aim of this paper is to prove the existence of the problem $(QES)_{f,g}$ in the space $W_0^{1,P}(\Omega,\omega,\mathbb{R}^m)$.

Remark 2.1 *-The condition $(F_0)^*$ and (G_0) ensure the measurability of f and g for all measurable function u.*

- *(F_1) and $(G_1)^*$ ensure that growths conditions, in particularly: if $u\in W_0^{1,P}(\Omega,\omega,\mathbb{R}^m)$ then $f(.,u(.),D(.))\cdot u(.)$ and $g(.,u):Du$ is in $L^1(\Omega,\omega)$.*
- *Exploiting the convergence in measure of the gradients of the approximating solutions, we will prove the following theorem.*

Theorem 2.4 *If $p\in(1,\infty)$ and σ satisfies the conditions (H_0)-(H_3), then the Dirichlet problem $(QES)^*_{f,g}$ has a weak solution $u\in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, for every $v\in W^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m)$, f satisfies $(F_0)^*$ and $(F_1)^*$ and g satisfies (G_0) and (G_1).*

In order to prove theorems, we will apply a Galerkin scheme, with this aim in view, we establish in the following subsections, the key ingredient to pass to the limit for this, we assume that the conditions: (H_0)-(H_3), $(F_0)^*$, $(F_1)^*$, (G_0) and (G_1).

Lemma 2.1 *For arbitrary $u\in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ and $v\in W^{-1,p'}(\Omega,\omega^*,\mathbb{R}^m)$, the functional*

$$\begin{aligned}F(u):W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)&\to\mathbb{R}\\ \varphi\mapsto&\int_\Omega\sigma(x,u(x),Du(x)):D\varphi(x)\mathrm{d}x-\langle v,\varphi\rangle\\ &-\int_\Omega f(x,u,Du):\varphi\mathrm{d}x+\int_\Omega g(x,u):D\varphi\mathrm{d}x.\end{aligned}$$

is well defined, linear and bounded.

Proof For all $\varphi\in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, we denote

$$F(u)(\varphi)=I_1+I_2+I_3+I_4$$

with

$$I_1=\int_\Omega\sigma(x,u(x),Du(x)):D\varphi(x)\mathrm{d}x,$$

and

$$I_2=-\langle v,\varphi\rangle.$$

$$I_3=-\int_\Omega f(x,u,Du):\varphi\mathrm{d}x$$

$$I_4=\int_\Omega g(x,u):D\varphi\mathrm{d}x$$

We define

$$I_{rs}=\int_\Omega\sigma_{rs}(x,u(x),Du(x)):D_{rs}\varphi(x)\mathrm{d}x$$

Firstly, by virtue of the growth conditions (H_2) and the Hölder inequality, one has

$$\begin{aligned}|I_{rs}| &\le \int_\Omega \left|\sigma_{rs}\left(x,u(x),Du(x)\right)\right| : \left|D_{rs}\varphi(x)\right| \mathrm{d}x \\ &\le \int_\Omega \beta\omega_{rs}^{1/p}(x)\left[\lambda_1(x)+c_1\sum_{j=1}^{m}\left|\gamma_j(x)\right|^{1/p'}\left|u_j(x)\right|^{q/p'}\right. \\ &\quad \left. +c_1\sum_{1\le i\le n,1\le j\le m}\omega_{ij}^{1/p'}\left|D_{ij}u\right|^{p-1}\right]\left|D_{rs}\varphi\right|\mathrm{d}x \\ &\le \beta\left[\left(\int_\Omega\left|\lambda_1(x)\right|^{p'}\mathrm{d}x\right)^{1/p'}\left(\int_\Omega\left|D_{rs}\varphi(x)\right|^{p}\omega_{rs}\mathrm{d}x\right)^{1/p}\right. \\ &\quad +\left(\int_\Omega\left|D_{rs}\varphi(x)\right|^{p}\omega_{rs}\right)^{1/p}\left(\sum_{j=1}^{m}\int_\Omega\left|u_j\right|^{q}\gamma_j\mathrm{d}x\right)^{1/p'} \\ &\quad \left.+\left(\sum_{1\le i\le n,1\le j\le m}\int_\Omega\left|D_{ij}u\right|^{p}\omega_{ij}\mathrm{d}x\right)^{1/p'}\left(\int_\Omega\left|D_{rs}\varphi\right|^{p}\omega_{rs}\mathrm{d}x\right)^{1/p}\right]\end{aligned}$$

with $\left(p=p'(p-1)\right)$, and thanks to Hardy inequality we have:

$$\begin{aligned}|I_{rs}| &\le c\beta\left[\left\|\lambda_1\right\|_{p'}\left\|\varphi\right\|_{1,p,\omega_{rs}}+c_1\left\|D\varphi\right\|_{p,\omega_{rs}}\left(\int_\Omega|u|^{q}\gamma\mathrm{d}x\right)^{1/p'}+c_1\sum_{ij}\left\|D\varphi\right\|_{p,\omega_{ij}}\left\|Du\right\|_{p,\omega_{rs}}\right] \\ &\le c'\beta\left[\left\|\lambda_1\right\|_{p'}\left\|\varphi\right\|_{1,p,\omega_{rs}}+\left\|\varphi\right\|_{1,p,\omega_{rs}}\left\|u\right\|_{q,\gamma}+\left\|u\right\|_{1,p,\omega}\left\|\varphi\right\|_{1,p,\omega_{rs}}\right]\end{aligned}$$

with $c'=\max(c,1)$. Which gives

$$|I_1|\le c'\beta\left[\left\|\lambda_1\right\|_{p'}+\left\|u\right\|_{1,p,\omega}^{q/p'}+\left\|u\right\|_{1,p,\omega}\right]\left\|\varphi\right\|_{1,p,\omega}<\infty.$$

and

$$|I_2|\le\int_\Omega|v||\varphi|\mathrm{d}x\le\|v\|_{-1,p',\omega^*}\|\varphi\|_{1,p,\omega}<\infty.$$

$$I_3=\sum_j\int_\Omega f_j(x,u,Du)\varphi_j(x)\mathrm{d}x$$

We denote $I_{3,j}=\left|\int_\Omega f_j(x,u,Du)\varphi_j(x)\mathrm{d}x\right|$.

$$\begin{aligned}I_{3,j} &\le \int_\Omega\left|f_j(x,u,Du)\right|\left|\varphi_j(x)\right|\mathrm{d}x \\ &\le \int_\Omega b_1(x)\left|\varphi_j(x)\right|\omega_{0j}^{\frac{1}{P}}\mathrm{d}x+c_1'\int_\Omega\gamma_j^{\frac{1}{p'}}\left|u_j\right|^{\frac{q}{p'}}\left|\varphi_j(x)\right|\omega_{0j}^{\frac{1}{P}} \\ &\quad +c_2'\int_\Omega\sum_{rs}\left|\varphi_j(x)\right|\left|D_{rs}u\right|^{p-1}\omega_{0j}^{\frac{1}{P}}\mathrm{d}x \\ &\le \left(\int_\Omega\left|b_1(x)\right|^{p'}\right)^{\frac{1}{p'}}\left(\int_\Omega\left|\varphi_j(x)\right|^{p}\omega_{0j}\mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad +\left(\int_\Omega\gamma_j(x)\left|u_j\right|^{q}\mathrm{d}x\right)^{\frac{1}{p'}}\left(\int_\Omega\left|\varphi_j(x)\right|^{p}\omega_{0j}\mathrm{d}x\right)^{\frac{1}{p}} \\ &\quad +\sum_{rs}\left(\int_\Omega\left|\varphi_j\right|^{p}\omega_{0j}\right)^{\frac{1}{p}}\cdot\left(\int_\Omega\left|D_{rs}u\right|^{p'(p-1)}\omega_{rs}\right)^{\frac{1}{p'}} \\ &\le \left\|b_1\right\|_{p'}\left\|\varphi\right\|_{1,p,\omega}+c_1'\left(\sum_j\int_\Omega\gamma_j(x)\left|u_j\right|^{q}\mathrm{d}x\right)^{\frac{1}{p'}}\cdot\left\|\varphi\right\|_{1,p,\omega}+c_2'\left\|\varphi\right\|_{1,\omega,p}\cdot\left\|Du\right\|_{1,p,\omega}^{\frac{p}{p'}}\end{aligned}$$

$$\leq \|b_1\|_{p'} \|\varphi\|_{1,p,\omega} + c_1' \cdot \|Du\|_{1,p,\omega} \cdot \|\varphi\|_{1,p,\omega} + c_2' \|\varphi\|_{1,\omega,p} \cdot \|Du\|_{1,p,\omega}^{\frac{p}{p'}}$$

$$\leq \left(\|b_1\| + c_1' \cdot \|Du\|_{1,p,\omega} + c_2' \cdot \|Du\|_{1,p,\omega}^{\frac{p}{p'}} \right) \cdot \|\varphi\|_{1,p,\omega}.$$

$$I_4 = \sum_{rs} \int_\Omega g_{rs}(x,u) D_{rs}\varphi \mathrm{d}x$$

$$\int_\Omega |g_{rs}| : |D_{rs}\varphi| \mathrm{d}x$$

$$\leq \int_\Omega b_2 \omega_{rs}^{\frac{1}{p}} \cdot D_{rs}\varphi \mathrm{d}x + \sum_j \int_\Omega \gamma_j^{\frac{1}{p'}}(x) |u_j|^{\frac{q}{p'}} \omega_{rs}^{\frac{1}{p}} D_{rs}\varphi \mathrm{d}x$$

$$\leq \left(\int_\Omega |b_2|^{p'} \mathrm{d}x \right)^{\frac{1}{p'}} \cdot \left(\int_\Omega |D_{rs}\varphi|^p \omega_{rs} \mathrm{d}x \right)^{\frac{1}{p}}$$

$$+ \sum_j \left(\int_\Omega |u_j|^q \gamma_j(x) \mathrm{d}x \right)^{\frac{1}{p'}} \left(\int_\Omega |D_{rs}\varphi|^p \omega_{rs}(x) \mathrm{d}x \right)^{\frac{1}{p}}$$

$$\leq \|b_2\|_{p'} \|D_{rs}\varphi\|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}} \left(\int_\Omega |D_{rs}\varphi|^p \omega_{rs} \mathrm{d}x \right)^{\frac{1}{p}}$$

$$I_4 \leq \|b_2\|_{p'} \|D_{rs}\varphi\|_{1,p,\omega_{rs}} + \|u\|_{q,\gamma}^{\frac{q}{p'}} \left(\int_\Omega |D_{rs}\varphi|^p \omega_{rs} \mathrm{d}x \right)^{\frac{1}{p}}$$

$$\leq \|b_2\|_{p'} \cdot \|D\varphi\|_{1,p,\omega} + \|u\|_{q,\gamma}^{\frac{q}{p'}} \cdot \|D\varphi\|_{1,p,\omega}$$

$$\leq c'' \|\varphi\|_{1,p,\omega}$$

Hence $I \leq c_4 \|\varphi\|_{1,p,\omega}$. With $c_4 < \infty$.

Finally the functional $F(.)$ is bounded.

Lemma 2.2 *The restriction of F to a finite dimensional linear subspace V of* $W_0^{1,p}(\Omega, \omega, I\!R^m)$ *is continuous.*

Proof Let d be the dimension of V and $(e_1, e_2, \cdots, e_d)$ a basis of *V*. Let $u_j = \sum_{1 \leq i \leq d} a_j^i \cdot e_i$ be a sequence in V which converges to $u = \sum_{1 \leq i \leq d} a^i e_i$ in *V*. The sequence (a_j) converge to $a \in I\!R^d$, so $u_j \to u$ and $Du_j \to Du$ a.e., on the other hand $\|u_j\|_p$ and $\|Du_j\|_p$ are bounded by a constant *c*. Thus, it follows by the continuity conditions (H_1), that

$$\sigma(x, u_j, Du_j) : D\varphi \to \sigma(x, u, Du) : D\varphi$$

for all $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$ and a.e. in Ω. Let Ω' be a measurable subset of Ω and let $\varphi \in W_0^{1,p}(\Omega, \omega, I\!R^m)$.

Thanks to the condition (H_2), we get

$$\int_{\Omega'} \left| \sigma(x, u_j, Du_j) : D\varphi \right| \mathrm{d}x < \infty,$$

By the continuity conditions $(F_0)^*$ and (G_0) we have:

$$f(x, u_j, Du_j) \cdot \varphi \to f(x, u, Du) \cdot \varphi$$

And

$$g(x, u_j) \cdot D\varphi \to g(x, u) \cdot D\varphi$$

almost everywhere. Moreover we infer from the growth conditions $(F_1)^*$ and (G_1) that the sequences:

$\left(\sigma\left(x,u_j,Du_j\right):D\varphi\right)$, $\left(f\left(x,u_j,Du_j\right)\cdot\varphi\right)$ and $\left(g\left(x,u_j\right)\cdot D\varphi\right)$

Are equi-integrable. Indeed, if $\Omega'\subset\Omega$ is a measurable subset and $\varphi\in W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$ then:

$\int_{\Omega'}\left|f\left(x,u_j,Du_j\right)\cdot\varphi\right|\mathrm{d}x<\infty$ (by $(F_1)^*$ and Hölder inequality).

$\int_{\Omega'}\left|g\left(x,u_j\right)\cdot D\varphi\right|\mathrm{d}x<\infty$ (by (G_1) and Hölder inequality).

$\int_{\Omega'}\left|\sigma\left(x,u_j,Du_j\right):D\varphi\right|\mathrm{d}x<\infty$ (by Hölder inequality).

which implies that $\sigma\left(x,u_j,Du_j\right):D\varphi$ is equi-integrable. And by applying the Vitali's theorem, it follows that

$$\int_\Omega\sigma\left(x,u_j,Du_j\right):D\varphi\mathrm{d}x\to\int_\Omega\sigma\left(x,u,Du\right):D\varphi\mathrm{d}x,$$

for all $\varphi\in W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$.

Finally

$$\lim_{j\to\infty}\left\langle F\left(u_j\right),\varphi\right\rangle=\left\langle F\left(u\right),\varphi\right\rangle,$$

which means that

$$F\left(u_j\right)\to F\left(u\right)\text{ in }W^{-1,p'}\left(\Omega,\omega^*,I\!R^m\right).$$

Remark 2.2 *Now, the problem* $(QES)_{f,g}^*$ *is equivalent to find a solution* $u\in W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$ *such that* $\left\langle F\left(u\right),\varphi\right\rangle=0$, *for all* $\varphi\in W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$.

In order to find such a solution we apply a Galerkin scheme.

3. Galerkin Approximation

Remark 3.1 (*Galerkin Schema*)

Let $V_1\subset V_2\subset\cdots\subset W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$ *be a sequence of finite dimensional subspaces with* $\bigcup_{k\in I\!N}V_k$ *dense in* $W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$. *The sequence* V_k *exists since* $W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$ *is separable.*

Let us fix some *k*, we assume that V_k has a dimension *d* and that $(e_1,e_2,\cdots,e_d)$ is a basis of V_k. Then, we define the map,

$$G:I\!R^k\to I\!R^k$$
$$(a_1,\cdots,a_k)\mapsto\left(\left\langle F(u),e_1\right\rangle,\cdots,\left\langle F(u),e_k\right\rangle\right);u=\sum_{i=1}^d a_ie_i.$$

Proposition 3.1 *The map G is continuous and* $G(a)\cdot a$ *tends to infinity when* $\|a\|_{I\!R^k}$ *tends to infinity.*

Proof. Since *F* restricted to V_k is continuous by Lemma 2.2, so *G* is continuous.

Let $a\in I\!R^d$ and $u=\sum_{1\le i\le d}a^i\cdot e_i$ in V_k, then $G\left(a\right)\cdot a=\left\langle F\left(u\right),u\right\rangle$ and which implies that $\|a\|_{I\!R^d}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

$$G\left(a\right)\cdot a=\sum_{1\le i\le d}\left\langle F\left(u\right),a^i\cdot e_i\right\rangle=\left\langle F\left(u\right),u\right\rangle$$

and

$$\|u\|_{1,p,\omega}^{p} = \left\| \sum_{1\le i\le d} a^i \cdot e_i \right\|_{1,p,\omega}^{p} \le \left(\sum_{1\le i\le d} |a^i| \cdot \|e_i\|_{1,p,\omega} \right)^p$$
$$\le \max_{1\le i\le d} \left(\|e_i\|_{1,p,\omega}^p \right) \cdot \left(\sum_{1\le i\le d} |a^i| \right)^p$$
$$\le c \cdot \|a\|_{\mathbb{R}^p} ,$$

which implies that $\|a\|_{\mathbb{R}^p}$ tends to infinity if $\|u\|_{1,p,\omega}$ tends to infinity.

Now, it suffices to prove that

$$\langle F(u), u \rangle \to \infty \quad \text{when } \|u\|_{1,p,\omega} \to \infty.$$

Indeed, thanks to the first coercivity condition and the Hölder inequality, we obtain

$$I = \int_\Omega \sigma(x,u,Du) : Du \mathrm{d}x$$
$$\ge -\|\lambda_2\|_1 - \int_\Omega \lambda_3 \omega_{0j}^{\alpha/p} |u_j|^\alpha \mathrm{d}x + c_2 \sum_{1\le i,j\le n,m} \int_\Omega |D_{ij}u|^p \omega_{ij} \mathrm{d}x$$

By the Hölder inequality, we have

$$\int_\Omega \lambda_3 |u_j|^\alpha \omega_{0j}^{\alpha/p} \mathrm{d}x \le \|\lambda_3\|_{(p/\alpha)'} \left(\int_\Omega \omega_{0j} |u_j|^{(p/\alpha)\cdot\alpha} \right)^{\alpha/p}$$
$$\le c' \|\lambda_3\|_{(p/\alpha)'} \|u_j\|_{1,p,\omega_{0j}} .$$

where c' is a constant positive. For $\|u\|_{1,p,\omega}$ large enough, we can write

$$|I| \ge -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u_j\|_{1,p,\omega_{0j}}^{\alpha} + c_2 \cdot \sum\nolimits_{1\le i,j\le n,m} \|Du_j\|_{1,p,\omega_{ij}}^p$$
$$\ge -\|\lambda_2\|_1 - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|_{1,p,\omega}^{\alpha} + c_2 c' \cdot \|u\|_{1,p,\omega}^{p} .$$

And since

$$|I'| = |\langle v,u \rangle| \le \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega}$$

Finally, it follows from the growth condition $(F_1)^*$ and G_1 that:

$$|I''| = \left| \int_\Omega f(x,u,Du) \cdot u \mathrm{d}x \right|$$
$$\le \left(\|b_1\|_{p'} + c_1' \cdot \|Du\|_{1,p,\omega} + c_2' \|Du\|_{1,p,\omega} \right) \cdot \|u\|_{1,p,\omega}$$
$$\le c_3 \cdot \|u\|_{1,p,\omega}$$

$$|I'''| = \left| \int_\Omega g(x,u) \cdot Du \mathrm{d}x \right| \le \left(\|b_2\|_{p'} + \|u\|_{q,\gamma}^{\frac{q}{p'}} \right) \cdot \|Du\|_{1,p,\omega} \le c_4 \cdot \|u\|_{1,p,\omega} ;$$

with c_4 is a constant. With; $0 < \alpha < p$ and $p > 1$, we get:

$$I - I' - I'' \ge c_2 \cdot c' \cdot \|u\|_{1,p,\omega}^p - \|v\|_{-1,p',\omega^*} \cdot \|u\|_{1,p,\omega} - c' \|\lambda_3\|_{(p/\alpha)'} \cdot \|u\|_{1,p,\omega}^{\alpha} - \|\lambda_2\|_1 - c_3 . \|u\|_{1,p,\omega} \tag{3.1}$$

Consequently, by using (3.1), we deduce

$$I - I' - I'' \to \infty \quad \text{as } \|u\|_{1,p,\omega} \to \infty.$$

and

$$I''' \to \infty \ \text{ as } \|u\|_{1,p,\omega} \to \infty.$$

$$\langle F(u), u\rangle \to \infty \ \text{ as } \|u\|_{1,p,\omega} \to \infty.$$

Remark 3.2 *The properties of G allow us to construct our Galerkin approximations.*

Corollary 3.1 *For all* $k \in \mathbb{N}$, *there exists* $(u_k) \subset V_k$ *such that* $\langle F(u_k), \varphi\rangle = 0$, *for all* $\varphi \in V_k$.

Proof By the proposition 3.1, there exists $R > 0$, such that for all $a \in \partial B_R(0) \subset \mathbb{R}^d$, we have $G(a)\cdot a > 0$. And the usual topological argument see [Zei 86 proposition 2.8] [17] implies that $G(x) = 0$ has a solution $x \in B_R(0)$. So, for all $k \in \mathbb{I}n$, there exists $(u_k) \subset V_k$, such that

$$\langle F(x^j e_j), e_j\rangle = 0 \ \text{ for all } 1 \le j \le d, \text{ with } d = \dim V_k$$

Taking $u_k = (x_k^i e_i)$, $e_i \in V_k$, so we obtain:

$$\langle F(u_k), \varphi\rangle = 0, \text{ for all } \varphi \in V_k.$$

Proposition 3.2 *The Galerkin approximations sequence constructed in corollary* (3.1) *is uniformly bounded in* $W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$; *i.e.,*

There exists a constant $R > 0$, *such that* $\|u_k\|_{1,p,\omega} \le R$, *for all* $k \in \mathbb{N}$.

Proof Like in the proof of proposition (3.1), we can see that

$$\langle F(u), u\rangle \to \infty \ \text{ as } \|u\|_{1,p,\omega} \to \infty.$$

Then, there exists R satisfying $\langle F(u), u\rangle > 1$ when $\|u\|_{1,p,\omega} > R$. Now, for the sequence of Galerkin approximations $(u_k) \subset V_k$ of corollary (3.1), which satisfying $\langle F(u_k), u_k\rangle = 0$, we have the uniform bound $\|u_k\|_{1,p,\omega} \le R$, for all $k \in \mathbb{N}$.

Remark 3.3 *There exists a subsequence* (u_k) *of the sequence* $(u_k) \subset V_k$, *such that*:

$$u_k \rightharpoonup u \quad \textit{in } W_0^{1,p}(\Omega, \omega, \mathbb{R}^m)$$

and

$$u_k \to u \quad \textit{in measure in } L^r(\Omega, \mathbb{R}^m);$$

with

$$\begin{cases} 1 \le r \prec \dfrac{nps}{n(s+1) - ps} & \text{if } ps \prec n(s+1) \\ r \ge 1 & \text{if } n(s+1) \prec ps \end{cases}$$

The gradient sequence (Du_k) *generates the young measure* ϑ_x. *Since* $u_k \to u$ *in measure, then* (u_k, Du_k) *generates the Young measure* $(\delta_{u(x)} \otimes \vartheta_x)$, *see* [2]. *Moreover, for almost x in* Ω, *we have,*

1) ϑ_x *is the probability measure, i.e.,* $\|\vartheta_x\|_{mes} = 1$.
2) ϑ_x *is the* $W^{1,p,\omega}$ *gradient homogeneous young measure.*
3) $\langle \vartheta_x, id\rangle = Du(x)$, *see* [18].

Proof. See [2]. (Dolzmann, N. Humgerbuhler S. Muller, Non linear elliptic system …)

4. Passage to the Limit

Now, we are in a position to prove our main result under convenient hypotheses.

Let

$$I_k = \left(\sigma\left(x,u_k,Du_k\right)-\sigma\left(x,u,Du\right)\right):\left(Du_k - Du\right). \tag{4.1}$$

Lemma 4.1 (*Fatou lemma type*) (*See* [2]) *Let* $F:\Omega\times I\!R^m\times I\!M^{m\times n}\to I\!R$ *be a Carathéodory function, and* $u_k:\Omega\to I\!R^m$ *a measurable sequence, such that* (Du_k) *generates the Young measure* ϑ_x, *with* $\|\vartheta_x\|_{mes}=1$, *for a.e.* $x\in\Omega$. *Then*:

$$\liminf_{k\to\infty}\int_\Omega F\left(x,u_k,Du_k\right)\mathrm{d}x \ge \int_\Omega\int_{I\!M^{m\times n}} F\left(x,u,\zeta\right)\mathrm{d}\vartheta_x\left(\zeta\right)\mathrm{d}x, \tag{4.2}$$

which provided that the negative part of $F(x,u_k,Du_k)$ *is equi-integrable.*

Proof.

Lemma 4.2 *Let* $p>1$ *and* u_k *be a sequence which is uniformly bounded in* $W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$. *There exists a subsequence of* u_k (*for convenience not relabeled*) *and a function* $u\in W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$ *such that* $u_k\rightharpoonup u$ *in* $W_0^{1,p}\left(\Omega,\omega,I\!R^m\right)$

And such that $u_k\to u$ *in measure on* Ω *and in* $L^r\left(\Omega,I\!R^m\right)$, *with*:

$$\begin{cases} 1\le r\prec \dfrac{nps}{n(s+1)-ps} & \text{if } ps\prec n(s+1) \\ r\ge 1 & \text{if } n(s+1)\prec ps \end{cases}$$

Proof. see [10].

Lemma 4.3 *The sequence* (I_k) *is equi-integrable.*

Proof

We have

$$\begin{aligned} I_k &= \left(\sigma\left(x,u_k,Du_k\right)-\sigma\left(x,u,Du\right)\right):\left(Du_k-Du\right) \\ &= \left[\sigma\left(x,u_k,Du_k\right):Du_k\right]-\left[\sigma\left(x,u_k,Du_k\right):Du\right] \\ &\quad -\left[\sigma\left(x,u,Du\right):Du_k\right]+\left[\sigma\left(x,u,Du\right):Du\right] \\ &= I_k^1+I_k^2+I_k^3+I_k^4 \end{aligned} \tag{4.3}$$

We denote $\left(I_k^1\right)^- = -\left[\sigma\left(x,u_k,Du_k\right):Du_k\right]^-$. Thanks to the coercivity condition (H_2), we have

$$\begin{aligned} \int_{\Omega'}\left|\left(I_k^1\right)^-\right|\mathrm{d}x &\le \int_\Omega |\lambda_2| + c_2\sum_{1\le j\le m}\omega_{0j}^{\frac{\alpha}{p}}|\lambda_3|\cdot\left|u_{kj}\right|^\alpha + c\sum_{1\le i,j\le n,m}\omega_{ij}\left|D_{ij}u_k\right|^p\mathrm{d}x \\ &\le \|\lambda_2\|_1 + \int_{\Omega'}\left(\sum_{1\le j\le m}\omega_{0j}^{\alpha/p}\left|u_{kj}\right|^\alpha\right)^{p/\alpha}\|\lambda_3\|_{(p/\alpha)'} + c_2\|u_k\|_{1,\omega,p}^p \end{aligned} \tag{4.4}$$

with $p/\alpha\ge 1$. Therefore,

$$\begin{aligned} \int_{\Omega'}\left|\left(I_k^1\right)^-\right|\mathrm{d}x &\le \|\lambda_2\|_1 + \left(\sum_{1\le j\le m}\omega_{0j}\left|u_{kj}\right|^p\right)^{\alpha/p}\|\lambda_3\|_{(p/\alpha)'} + c_2\|u_k\|_{1,\omega,p}^p \\ &\le \|\lambda_2\|_1 + \|u_k\|_{p,\bar{\omega}_{00}}^\alpha\|\lambda_3\|_{(p/\alpha)'} + c_2\|u_k\|_{1,\omega,p}^p \\ &< \infty, \end{aligned}$$

for all $\Omega' \subset \Omega$.

Similarly for $\left|\left(I_k^4\right)^-\right|$.

Now, by using the growth condition (H_2) and the Hardy inequality (H_0), we have

$$\begin{aligned}
&\int_{\Omega'}\left|\left(I_k^2\right)^-\right|\mathrm{d}x=\int_{\Omega'}\left|\sigma\left(x,u_k,Du_k\right):Du_k\right|\mathrm{d}x\\
&\le\beta\int_{\Omega'}\omega_{rs}^{1/p}\left(\lambda_1+c_1\sum_{1\le j\le m}\gamma_j^{1/p'}\cdot\left|u_{kj}\right|^{q/p'}+c_2\sum_{1\le i,j\le n,m}\omega_{ij}^{1/p'}\left|D_{ij}u_k\right|^{p-1}\right)D_{rs}u_k\mathrm{d}x.
\end{aligned}\tag{4.5}$$

Thus, by the Hölder inequality, we obtain

$$\begin{aligned}
\int_{\Omega'}\left|\left(I_k^2\right)^-\right|\mathrm{d}x\le\beta\Bigg[&\left\|\lambda_1\right\|_{p'}\left(\int_{\Omega'}\left|D_{rs}u_k\right|^p\omega_{rs}\mathrm{d}x\right)^{1/p}\\
&+c_1\left(\int_{\Omega'}\left|D_{rs}u_k\right|^p\omega_{rs}\mathrm{d}x\right)^{1/p}\left(\int_{\Omega'}\left(\sum_{1\le j\le m}\gamma_j^{1/p'}\left|u_{kj}\right|^{q/p'}\right)^{p'}dx\right)^{1/p'}\\
&+c_1\left(\sum_{1\le j\le m}\int_{\Omega'}\left(\left|D_{ij}u_k\left(x\right)\right|^{p'(p-1)}\omega_{ij}\mathrm{d}x\right)^{1/p'}\right)\left(\int_{\Omega'}\left|D_{rs}u_k\right|^p\omega_{rs}\mathrm{d}x\right)^{1/p}\Bigg].
\end{aligned}\tag{4.6}$$

So, by combining (4.5) and (4.6), we deduce that

$$\int_{\Omega'}\left|\sigma\left(x,u_k,Du_k\right):Du_k\right|\mathrm{d}x\le c'\beta\left(\left\|\lambda_1\right\|_{p'}\left\|u_k\right\|_{1,p,\omega}+\left\|u_k\right\|_{1,p,\omega}\right)<\infty.\tag{4.7}$$

Similarly to $\left|\left(I_k^2\right)^-\right|$, we obtain $\left|\left(I_k^3\right)^-\right|$. Finally: I_k is equi-integrable.

We choose a sequence φ_k such that φ_k belongs to the same space V_k and $\varphi_k\to\varphi$ in $W_0^{1,p}\left(\Omega,\omega,\mathbb{R}^m\right)$, this allows us in particular, to use $u_k-\varphi_k$ as a test function in (3.1). We have:

$$\begin{aligned}
&\int_{\Omega}\left|\sigma\left(x,u_k,Du_k\right):\left(Du_k-D\varphi_k\right)\right|\mathrm{d}x\\
&=\left\langle v,u_k-\varphi_k\right\rangle+\int_{\Omega}f\left(x,u_k,Du_k\right)\cdot\left(u_k-\varphi_k\right)\mathrm{d}x\\
&\quad-\int_{\Omega}g\left(x,u_k\right):\left(Du_k-D\varphi_k\right)\mathrm{d}x.
\end{aligned}\tag{4.8}$$

The first term on the right in 4.8 converge to zero since $\left(u_k-\varphi_k\right)\rightharpoonup 0$ in $W_0^{1,p}\left(\Omega,\omega,\mathbb{R}^m\right)$. By the choice of φ_k, the sequence φ_k uniformly bounded in $W_0^{1,p}\left(\Omega,\omega,\mathbb{R}^m\right)$, and lemma (4.2). Next, for the second term: $II_k=\int_{\Omega}f\left(x,u_k,Du_k\right)\cdot\left(u_k-\varphi_k\right)\mathrm{d}x$ in 4.8 it follows from the growth condition F_1^* and the Hölder inequality that:

$$\begin{aligned}
\left|II_k\right|&\le\left(\left\|b_1\right\|_{p'}+c_1'\cdot\left\|D\left(u_k-\varphi_k\right)\right\|_{1,p,\omega}+c_2'\cdot\left\|D\left(u_k-\varphi_k\right)\right\|^{\frac{p}{p'}}\right)\cdot\left\|u_k-\varphi_k\right\|_{1,p,\omega}\\
&\le\left(\left\|b_1\right\|_{p'}+c\cdot\left\|D\left(u_k-\varphi_k\right)\right\|_{1,p,\omega}\right)\cdot\left\|u_k-\varphi_k\right\|_{1,p,\omega}
\end{aligned}$$

By the equivalence of the norm in $W_0^{1,p}\left(\Omega,\omega,\mathbb{R}^m\right)$ and the sequence u_k is uniformly bounded in $W_0^{1,p}\left(\Omega,\omega,\mathbb{R}^m\right)$, $\left\|u_k\right\|_{1,p,\omega}$ is bounded.

Moreover, by the construction of φ_k, and lemma (4.2) we have:

$$\left\|u_k-\varphi_k\right\|_{1,p,\omega}\le\left\|u_k-u\right\|_{1,p,\omega}+\left\|u-\varphi_k\right\|_{1,p,\omega}$$

$$\left(\left\|u_k-u\right\|_{1,p,\omega}+\left\|u-\varphi_k\right\|_{1,p,\omega}\right)\to 0$$

We infer that the second term in 4.8 vanishes as $k \to \infty$. Finally, for the last term

$$III_k = \int_\Omega g(x,u_k) : D(u_k - \varphi_k)\mathrm{d}x$$

in 4.8, we note that

$$g(x,u_k) \to g(x,u)$$

Strongly in $L^{p'}(\Omega, M^{m\times n})$ by (G_0), (G_1) and lemma (4.2).

Indeed we may assure that $u_k \to u$ almost everywhere.

$$\begin{aligned} III_k &\le \left(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \right) \cdot \|D(u_k - \varphi_k)\|_{1,p,\omega} \\ &\le c' \cdot \left(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \right) \cdot \|(u_k - \varphi_k)\|_{1,p,\omega} \\ &\le c' \cdot \left(\|b_2\|_{p'} + \|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \right) \cdot \left(\|u_k - u\|_{1,p,\omega} + \|\varphi_k - u\|_{1,p,\omega} \right) \end{aligned}$$

$\|\varphi_k - u\|_{1,p,\omega} \to 0$, $\|u_k - u\|_{1,p,\omega} \to 0$ and $\|u_k - \varphi_k\|_{q,\gamma}^{\frac{q}{p'}} \to 0$

Now, we consider $(I_k)' = \sigma(x,u_k,Du_k) : (Du_k - Du)$. We have, I_k' is equi-integrable because I_k it is. So, we define

$$\begin{aligned} X &= \liminf \int_\Omega I_k \mathrm{d}x = \liminf \int_\Omega (I_k)' \,\mathrm{d}x \\ &\ge \int_\Omega \int_{IM^{m\times n}} \left(\sigma(x,u,\lambda) : (\lambda - Du) \right) \mathrm{d}\vartheta_x(\lambda) \end{aligned}$$

So to prove (??), it suffices to prove that:

$$X \le 0. \tag{4.9}$$

Let $\varepsilon > 0$, so there exists $k_0 \in IN$ such that, for all $k > k_0$, we have $dist(u,V_k) < \varepsilon$ since: $\liminf_{\varphi_k \in V_k} \|u - \varphi_k\|_{1,p,\omega} < \varepsilon$, $(u_k \rightharpoonup u)$

Or in an equivalent manner $dist(u_k - u, V_k) < \varepsilon$, $\forall k > k_0$ then for all $v_k \in V_k$, we have

$$\begin{aligned} X &= \liminf_{k\to\infty} \int_\Omega \left(\sigma(x,u_k,Du_k) : (Du_k - Du) \right) \mathrm{d}x \\ &= \liminf_{k\to\infty} \left[\int_\Omega \left(\sigma(x,u_k,Du_k) : D(u_k - u - \varphi_k) \right) \mathrm{d}x + \int_\Omega \left(\sigma(x,u_k,Du_k) : D(\varphi_k) \right) \right] \end{aligned}$$

Combining (H_2) and (0.1), we get

$$\begin{aligned} X \le \lim\inf_{k\to\infty} \int_\Omega \beta \omega_{rs}^{1/p} &\left[\lambda_1 + c_1 \sum_{1\le j\le m} \gamma_j^{1/p'} \left| u_{k_j} \right|^{q/p'} + c_1 \sum_{1\le i,j\le n,m} \omega_{ij}^{1/p'} \left| D_{ij} u_k \right|^{p-1} \right] \\ &\times \left| D_{rs}(u_k - u - \varphi_k) \right| \mathrm{d}x + \langle v, \varphi_k \rangle. \end{aligned}$$

For all $\varepsilon > 0$, we choose $\varphi_k \in V_k$ such that

$$\|u_k - u - \varphi_k\|_{1,p,\omega} \le 2\varepsilon, \tag{4.10}$$

For all $k \ge k_0$, which implies that

$$\left| \langle v, \varphi_k \rangle \right| \le \left| \langle v, \varphi_k + (u - u_k) \rangle \right| + \left| \langle v, u_k - u \rangle \right| \le 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k)$$

Hence $\lim_{k\to\infty} \langle v, u_k - u \rangle = 0$. According to Hölder and Hardy inequalities, and by (4.1) we deduce that

$$X \le \liminf_{k\to\infty} c\beta\Bigg(\|\lambda_1\|_{p'} \left(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p \cdot \omega_{rs}\mathrm{d}x\right)^{1/p} + c_1\left(\int_\Omega |u_k|^q \cdot \gamma\right)^{1/p'} \cdot \left(\int_\Omega |D_{rs}(u_k - u - \varphi_k)|^p \omega_{rs}\mathrm{d}x\right)^{1/p} + c_1\left(\sum \int_\Omega \omega_{ij} |D_{ij}u|^{p'(p-1)}\right)^{1/p'} \cdot \left(\int_\Omega \omega_{rs} |D_{rs}(u_k - u - \varphi_k)|^p\right)^{1/p}\Bigg) + |\langle v, \varphi_k\rangle|$$

$$\le \liminf_{k\to\infty} c\left(\|\lambda_1\|_{p'} \cdot \|u_k - u - \varphi_k\|_{1,p,\omega}\right) + \|u_k\|^q_{1,p,\omega} \|u_k - u - \varphi_k\|_{1,p,\omega} + 2\varepsilon \|v\|_{-1,p',\omega^*} + o(k)$$

Therefore,

$$X \le 2\varepsilon c\beta\left(\|\lambda_1\|_{p'} + \|u\|^q_{1,p,\omega} + \|v\|_{-1,p',\omega^*}\right).$$

which proves that $X \le 0$, and finally

$$\int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda) : \lambda \mathrm{d}\vartheta_x \mathrm{d}x \le \int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda) : Du \mathrm{d}\vartheta_x(\lambda)\mathrm{d}x.$$

Proof of theorem:

For arbitrary φ in $W_0^{1,p}(\Omega,\omega,IR^m)$. It follows from the continuity condition (F_0^*) and (G_0) that

$$f(x,u_k,Du_k)\cdot\varphi(x) \to f(x,u,Du)\cdot\varphi(x)$$

and

$$g(x,u_k) : D\varphi(x) \to g(x,u) : D\varphi(x)$$

almost everywhere. Since, by the growth conditions (F_1^*), (G_1) and the uniform bound of u_k, $f(x,u_k,Du_k)\cdot\varphi(x)$ and $g(x,u_k) : D\varphi(x)$ are equi-integrable, it follows that the Vitali's theorem. This implies that:

$$\lim_{k\to\infty} \int_\Omega f(x,u_k,Du_k)\cdot\varphi(x)\mathrm{d}x = \int_\Omega f(x,u,Du)\cdot\varphi(x)\mathrm{d}x$$

for all $\varphi \in \bigcup_{k=1}^\infty V_k$ and

$$\lim_{k\to\infty} \int_\Omega g(x,u_k) : D\varphi(x)\mathrm{d}x = \int_\Omega g(x,u) : D\varphi(x)\mathrm{d}x$$

for all $\varphi \in \bigcup_{k=1}^\infty V_k$ We will start with the easiest case

(d): $F \mapsto \sigma(x,u,F)$ is strict p-quasi-monotone. (4.11)

Indeed, we assume that ϑ_x is not a Dirac mass on the set M with $x \in M$ of positive Lebesgue measure $|M| > 0$. Moreover, by the strict p-quasi-monotonicity of $\sigma(x,u,\cdot)$ and ϑ_x is an homogeneous $W^{1,p}$ gradient young measure for a.e. $x \in M$. So, for a.e. $x \in M$, with $\bar{\lambda} = \langle \vartheta_x, Id\rangle = apDu(x)$, with $apDu(x)$ is the differentiable approximation in *x*. We get

$$\int_{IM^{m\times n}} \sigma(x,u,\lambda) : (\lambda - Du)\mathrm{d}\vartheta_x(\lambda)$$
$$> \int_{IM^{m\times n}} \sigma(x,u,Du) : (\lambda - Du)\mathrm{d}\vartheta_x(\lambda)$$
$$> \sigma(x,u,Du) : \int_{IM^{m\times n}} \lambda \mathrm{d}\vartheta_x(\lambda) - \sigma(x,u,Du) : Du \int_{IM^{m\times n}} \mathrm{d}\vartheta_x(\lambda)$$
$$> \left(\sigma(x,u,Du) : Du - \sigma(x,u,Du) : Du\right) = 0$$
$$> 0$$

On the other hand (4.9), integrating over Ω, and using the div-cul inequality we have:

$$\begin{aligned}&\int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda):\lambda \mathrm{d}\vartheta_x(\lambda)\mathrm{d}x\\ &> \int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda):Du \mathrm{d}\vartheta_x(\lambda)\mathrm{d}x\\ &\geq \int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda):\lambda \mathrm{d}\vartheta_x(\lambda)\mathrm{d}x.\end{aligned}$$

Which is a contradiction with (3.8). Thus $\vartheta_x = \delta_{\bar{\lambda}} = \delta_{Du(x)}$ for a.e. $x \in \Omega$. Therefore, $Du_k \to Du$ in measure when k tends to infinity. Then, we get $\sigma(x,u_k,Du_k) \to \sigma(x,u,Du)$ for all $x \in \Omega$. In the other hand, for all $\varphi \in \bigcup_{k\in IN} \vartheta_k$; $\sigma(x,u_k,Du_k):D\varphi \to \sigma(x,u,Du):D\varphi$ a.e. $x \in \Omega$. Moveover, for all $\Omega' \subset \Omega$ measurable, it is easy to see that:

$$\int_{\Omega'} \sigma(x,u_k,Du_k):D\varphi \mathrm{d}x \leq c\beta\left(\|\lambda_1\|_{p'} + \|u_k\|_{1,p,\omega}^{q/p'} + \|u_k\|_{1,p,\omega}^{p/p'}\right)\|u\|_{1,p,\omega} < \infty,$$

because $\|u_k\|_{1,p,\omega} \leq R$. And thanks to Vitali's theorem, we obtain:

$$\langle F(u),\varphi\rangle = 0 \text{, for all } \varphi \in \bigcup_{k\in IN} \vartheta_k.$$

which proves the theorem in this case.

Remark 4.1 *Before treating the cases* (a), (b) *and* (c) *of* (H_3), *we note that*

$$\int_\Omega \int_{IM^{m\times n}} \left(\sigma(x,u,\lambda)-\sigma(x,u,Du)\right):(\lambda - Du)\mathrm{d}\vartheta_x(\lambda)\mathrm{d}x \leq 0 \tag{4.12}$$

Since

$$\int_\Omega \int_{IM^{m\times n}} \sigma(x,u,\lambda):(\lambda - Du)\mathrm{d}\vartheta_x(\lambda)\mathrm{d}x = 0,$$

thanks to the div-Curl inequality in (4.9). On the other hand, the integrand in (4.12) is non negative, by the monotonicity of σ. Consequently, the integrating should be null, a.e., with respect to the product measure $\mathrm{d}\vartheta_x \otimes \mathrm{d}x$, which mean

$$\left(\sigma(x,u,\lambda)-\sigma(x,u,Du)\right):(\lambda - Du) = 0 \text{ in } spt\vartheta_x. \tag{4.13}$$

Thus,

$$spt\vartheta_x \subset \left\{\lambda \in IM^{m\times n} / \left(\sigma(x,u,\lambda)-\sigma(x,u,Du)\right):(\lambda - Du) = 0\right\}. \tag{4.14}$$

Case c: We prove that, the map $F \mapsto \sigma(x,u,F)$ is strictly monotone, for all $x \in \Omega$ and for all $u \in IR^m$.

Sine σ is strict monotone, and according to (4.14),

$$spt\vartheta_x = \{Du\}, \text{ i.e, } \vartheta_x = \delta_{Du}, \quad \text{a.e. in } \Omega,$$

which implies that, $Du_k \to Du$ in measure. For the rest of our prove is similarly to case d.

Case b: We start by showing that for almost all $x \in \Omega$, the support of ϑ_x is contained in the set where W agrees with the supporting hyper-plane.

$$L = \left\{\left(\lambda, W(x,u,\bar{\lambda}) + \sigma(x,u,\bar{\lambda}):(\lambda - \bar{\lambda})\right)\right\} \text{ with } \bar{\lambda} = Du(x).$$

So, it suffices to prove that

$$spt\vartheta_x \subset K_x = \left\{\lambda \in IM^{m\times n} / W(x,u,\lambda) = W(x,u,\bar{\lambda}) + \sigma(x,u,\bar{\lambda}):(\lambda-\bar{\lambda})\right\} \quad (4.15)$$

If $\lambda \in spt\vartheta_x$, thanks to (4.14), we have

$$(1-t)\cdot(\sigma(x,u,Du)-\sigma(x,u,\lambda)):(Du-\lambda)=0, \text{ for all } t\in[0,1]. \quad (4.16)$$

On the other hand, since σ is monotone, for all $t\in[0,1]$ we have:

$$(1-t)\cdot(\sigma(x,u,Du+t\cdot(\lambda-Du))-\sigma(x,u,\lambda)):(Du-\lambda)\geq 0. \quad (4.17)$$

By subtracting (4.16) from (4.17), we get

$$(1-t)\left[\sigma(x,u,\bar{\lambda}+t(\lambda-\bar{\lambda}))-\sigma(x,u,\bar{\lambda})\right]:(\bar{\lambda}-\lambda)\geq 0, \quad (4.18)$$

for all $t\in[0,1]$. Doing the same by the monotonicity in (4.18), we obtain

$$(1-t)\left[\sigma(x,u,\bar{\lambda}+t(\lambda-\bar{\lambda}))-\sigma(x,u,\bar{\lambda})\right]:(\bar{\lambda}-\lambda)\leq 0. \quad (4.19)$$

Combining (4.18) and (4.19), we conclude that

$$(1-t)\left[\sigma(x,u,\bar{\lambda}+t(\lambda-\bar{\lambda}))-\sigma(x,u,\bar{\lambda})\right]:(\bar{\lambda}-\lambda)=0, \quad (4.20)$$

for all $t\in[0,1]$, and for all $\lambda\in spt\vartheta_x$.

Now, it follows from (4.19) that

$$\begin{aligned} W(x,u,\lambda) &= W(x,u,\bar{\lambda})+\left(W(x,u,\lambda)-W(x,u,\bar{\lambda})\right) \\ &= W(x,u,\bar{\lambda})+\int_0^1\left[\sigma(x,u,\bar{\lambda})+t(\lambda-\bar{\lambda})\right]:(\lambda-\bar{\lambda})\mathrm{d}t \\ &= W(x,u,\bar{\lambda})+\sigma(x,u,\bar{\lambda}):(\lambda-\bar{\lambda}) \end{aligned}$$

Witch prove (4.15).

Now, by the coercivity of W, we get

$$W(x,u,\lambda)\geq W(x,u,\bar{\lambda})+\sigma(x,u,\bar{\lambda}):(\lambda-\bar{\lambda}),$$

for all $\lambda\in IM^{m\times n}$. Therefore,

L is a supporting hyper-plane, for all

$$\lambda\in K_x. \quad (4.21)$$

Moveover, the mapping $\lambda\mapsto W(x,u,\lambda)$ is continuously differentiable, so we obtain

$$\sigma(x,u,\lambda)=\sigma(x,u,\bar{\lambda}), \text{ for all } \lambda\in K_x. \quad (4.22)$$

Thus,

$$\bar{\sigma}(x)=\int_{IM^{m\times n}}\sigma(x,u,\lambda)\mathrm{d}\vartheta_x(\lambda)=\sigma(x,u,\bar{\lambda}). \quad (4.23)$$

Now, we consider the Carathéodory function

$$g_v(x,u,\rho)=\left|(\sigma(x,u,\rho)-\bar{\sigma}(x)):D\varphi\right|,$$

and lets $g_k(x)=g_\varphi(x,u_k,Du_k)$ is equi-integrable. Thus, thanks to BALL's theorem, see [6] $g_k \rightharpoonup g$ weakly in $L^1(\Omega)$, and the weakly limit of g is given by

$$\begin{aligned} \bar{g}_\varphi(x) &= \iint_{IR^m\times IM^{m\times n}}\left|\sigma(x,\eta,\lambda)-\bar{\sigma}(x)\right|\mathrm{d}\delta_{u(x)}(\eta)\otimes\mathrm{d}\vartheta_x(\lambda) \\ &= \int_{spt\vartheta_x}\left|\sigma(x,u(x),\lambda)-\bar{\sigma}(x)\right|\mathrm{d}\vartheta_x(\lambda) \\ &= 0. \end{aligned}$$

According to (4.22) and (4.23), and since $g_k \geq 0$, it follow that $g_k \to 0$ strongly in $L^1(\Omega)$ by Fatou lemma, which gives

$$\lim_{k\to\infty}\int_\Omega \sigma(x,u_k,Du_k):D\varphi\mathrm{d}x = \int_\Omega \sigma(x,u,Du):D\varphi\mathrm{d}x.$$

Thus

$$\langle F(u),\varphi\rangle = 0,\ \ \forall\varphi\in\bigcup_{k\in I\!N} V_k.$$

This completes the proof of the case (b).

Case (a): In this case, on $spt\vartheta_x$, we affirm that,

$$\sigma(x,u,\lambda):M = \sigma(x,u,Du):M + (\nabla_F\sigma(x,u,Du):M):(Du-\lambda), \quad (4.24)$$

for all $M \in I\!M^{m\times n}$, where ∇_F is the derivative with respect to the third variable of σ and $\bar{\lambda} = Du(x)$.

Thanks to the monotonicity of σ, we have

$$(\sigma(x,u,\lambda)-\sigma(x,u,Du+tM)):(\lambda-Du-tM)\geq 0, \text{ for all } t\in I\!R.$$

By invoking (4.19), we obtain

$$\begin{aligned}&-\sigma(x,u,\lambda):(tM)\\ &\geq -\sigma(x,u,Du):(\lambda-Du)+\sigma(x,u,Du+tM):(\lambda-Du-t\cdot M).\end{aligned}$$

On the other hand, $F\mapsto\sigma(x,u,F)$ is a C^1 function, so

$$\sigma(x,u,Du+tM) = \sigma(x,u,Du)+\nabla_F(x,u,Du)\cdot(tM)+o(t).$$

Thus

$$\begin{aligned}&-\sigma(x,u,\lambda):(t\cdot M)\\ &\geq -\sigma(x,u,Du):(tM)+\nabla_F\sigma(x,u,Du)(t\cdot M):(\lambda-Du)+o(t),\end{aligned}$$

which gives

$$\begin{aligned}&-\sigma(x,u,\lambda):(t\cdot M)\\ &\geq t\left[\nabla_F\sigma(x,u,Du):(M):(\lambda-Du)-\sigma(x,u,Du):(M)\right]+o(t),\end{aligned}$$

t is arbitrary in (4.24).

Finally for all $\varphi\in\bigcup_{k\in I\!N}V_k$ the sequence $\sigma(x,u_k,Du_k):D\varphi$ is equi-integrable. Then, by the BALL's theorem, see [1] the weak limit is $\int_{spt\vartheta_x}\sigma(x,u,\lambda):D\varphi\mathrm{d}\vartheta_x(\lambda)$

By choosing $M = Du$ in (4.24), we obtain

$$\begin{aligned}&\int_{spt\vartheta_x}(Du-\lambda)(\sigma(x,u,\lambda):D\varphi):D\varphi\mathrm{d}\vartheta_x(\lambda)\\ &=\int_{spt\vartheta_x}\sigma(x,u,Du):D\varphi\mathrm{d}\vartheta_x(\lambda)+(\nabla_F\sigma(x,u,Du):D\varphi)^t\int_{spt\vartheta_x}(Du-\lambda)\mathrm{d}\vartheta_x(\lambda)\\ &=(\sigma(x,u,Du):D\varphi)\int_{spt\vartheta_x}\mathrm{d}\vartheta_x(\lambda)=\sigma(x,u,Du):D\varphi.\end{aligned}$$

Hence:

$$\sigma(x,u_k,Du_k):D\varphi\to\sigma(x,u,Du):D\varphi \text{ strongly}$$

This proves that

$$\langle F(u),\varphi\rangle = 0 \text{ for all } \varphi\in\bigcup V_k.$$

And since $\bigcup V_k$ is dense in $W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$, so u is a weak solution of $(QES)^*_{f,g}$, as desired.

Remark 4.2 *In case* (b) $\sigma(x,u_k,Du_k):D\varphi \to \sigma(x,u,Du):D\varphi$ *strongly, but in the case* (c) *and* (d) $Du_k \to Du$ *in measure.*

Exemple 4.1 *We shall suppose that the weight functions satisfy:*

$w_{i_0 j} = 0,\ j = 1,2,\cdots,m$ *for some* $i_0 \in I^c$; *and* $\omega_{ij}(x) = w(x); x \in \Omega$, *with* $I^c \bigcup I = \{0;1;2;\cdots;n\}$, *for all* $i \in I \sqcup I^c$, $j = 1,2,\cdots,m$, *and* $i \neq i_0$ *with* $w(x) > 0$ *a.e in* Ω *then, we can consider the Hardy inequality in the form:*

$$\left(\sum_{j=1}^{m}\int_\Omega |u_j(x)|^q \gamma_j(x)\mathrm{d}x\right)^{\frac{1}{q}} \le c\left(\sum_{1\le i\le N,1\le j\le m}\int_\Omega |D_{ij}u|^p \omega_{ij}\right)^{\frac{1}{p}},$$

for every $u \in W_0^{1,p}(\Omega,\omega,\mathbb{R}^m)$ *with a constant* $c > 0$ *independent of u and for some* $q > p'$. *Let us consider the Carathéodory functions:* $(\star)$

$$\sigma_{ij}(x,\eta,\xi_I) = \omega(x)|\xi_{ij}|^{p-1} sng(\xi_{ij}),\ j = 1,2,\cdots,m,\ i \in I$$

$$\sigma_{ij}(x,\eta,\xi_{I^c}) = \omega(x)|\xi_{ij}|^{p-1} sng(\xi_{ij}),\ j = 1,2,\cdots,m,\ i \in I^c,\ i \neq i_0$$

$$\sigma_{i_0 j}(x,\eta,\xi_{I^c}) = 0,\ j = 1,2,\cdots,m$$

$$f_j(x,\eta,\xi) = -sign(\xi)\sum_{rs}\omega_{rs}^{\frac{1}{p'}}|\xi|^{p-1}\omega_{0j}^{\frac{1}{p}}$$

The above functions defined by $(\star)$ *satisfies the growth conditions* (H_2).

In particular, let use the special weight function ω,γ *expressed in term of the distance to the boundary* $\partial\Omega$ *denote* $d(x) = dist(x;\partial\Omega)$ *and* $\omega(x) = d^\lambda(x)$, $\gamma_j(x) = d^\mu(x)$ *the hardy inequality reads:*

$$\left(\sum_{j=1}^{m}\int_\Omega |u_j(x)|^q \mathrm{d}^\mu(x)\mathrm{d}x\right)^{\frac{1}{q}} \le c\left(\sum_{1\le i\le N,1\le j\le m}\int_\Omega |D_{ij}u|^p \mathrm{d}^\lambda(x)\right)^{\frac{1}{p}},$$

and the corresponding $W_0^{1,p}(\Omega;\omega;R^m) \hookrightarrow L^q(\Omega;\gamma;R^m)$ *is compact if:*

1) For, $1 < p \le q < \infty$

$$\lambda < p-1; \frac{n}{q} - \frac{n}{p} + 1 \ge 0; \frac{\mu}{q} - \frac{\lambda}{p} + \frac{n}{q} - \frac{n}{p} + 1 > 0$$

2) For, $1 \le q < p < \infty$

$$\lambda < p-1; \frac{n}{q} - \frac{n}{p} + 1 \ge 0; \frac{\mu}{q} - \frac{\lambda}{p} + \frac{1}{q} - \frac{1}{p} + 1 > 0$$

3) For, $q > 1$

$\mu(q'-1) < 1$, *by the simple modifications of the example in* [11]. *Moreover, the monotonicity condition are satisfied:*

$$\sum_{ij}\left(\sigma_{ij}(x,\eta,\xi_I) - \sigma_{ij}(x,\eta,\xi'_I)\right)\left(\xi_{ij} - \xi'_{ij}\right)$$
$$= \omega(x)\sum_{ij}\left(|\xi_{ij}|^{p-1} sng(\xi_{ij}) - |\xi'_{ij}|^{p-1} sng(\xi'_{ij})\right)\left(\xi_{ij} - \xi'_{ij}\right) \ge 0$$

for almost all $x \in \Omega$ *and for all,* $\xi,\xi' \in M^n$. *This last inequality cannot be* *strict, since for* $\xi_{I^c} \neq \xi'_{I^c}$ *with* $\xi_{i_0 j} \neq \xi'_{i_0 j}$ *for all* $j = 1, 2, \cdots, m$. *But* $\xi_{ij} = \xi'_{ij}$ *for* $i \in I^c$, $i \neq i_0$, $j = 1, 2, \cdots, m$ *the corresponding expression is Zero.*

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

References

[1] Ball, J. (1989) A Version of the Fundamental Theorem for Young Measures. In Partial Differential Equations and Continuum Models of Phase Transitions. *Proceedings of an NSF-CNRS Joint Seminar*, Nice, France, 207-215.

[2] Dolzmann, G., Hungerbuhler, N. and Muller, S. (1997) Nonlinear Elliptic Systems with Measure-Valued Right Hand Side. *Mathematische Zeitschrift*, **226**, 545-574. https://doi.org/10.1007/PL00004354

[3] Minty, G.J. (1962) Monotone (Nonlinear) Operators in Hilbert Space. *Duke Mathematical Journal*, **29**, 341-346. https://doi.org/10.1215/S0012-7094-62-02933-2

[4] Brezis, H. (1973) Operateurs maximaux monotones et semi-groupes de contractions dans les espaces de Hilbert. North-Holland Mathematics Studies, No. 5. Notas de Matematica (50). North-Holland Publishing Co., Amsterdam-London, American Elsevier Publishing Co. Inc., New York.

[5] Edwin, H. and Stromberg, K. (1969) Real and Abstract Analysis. Springer-Verlag, Berlin.

[6] Fonseca, I., Muller, S. and Pedregal, P. (1998) Analysis of Concentration and Oscillation Effects Generated by Gradients. *SIAM Journal on Mathematical Analysis*, **29**, 736-756. https://doi.org/10.1137/S0036141096306534

[7] Lions, J.L. (1969) Quelques Methodes de Resolution des Problemes aux Limites non Lineaires. Dunod, Gauthier-Villars, Paris.

[8] Kristznsen, J. (1999) Lower Semicontinuity in Space of Weakly Differentiable Functions. *Mathematische Annalen*, **313**, 653-710. https://doi.org/10.1007/s002080050277

[9] Visik, M.I. (1963) Quasi-Linear Strongly Elliptic Systems of Differential Equations of Divergence Form. *Trudy Moskov. Mat. Obsc*, **12**, 125-184. (In Russian)

[10] Hungerbühler, N. (1999) Quasilinear Elliptic Systems in Divergence form with Weak Monotonicity. *New York Journal of Mathematics*, **5**, 83-90.

[11] Akdim, Y. and Azroul, E. (2002) Pseudo-Monotonicity and Degenerated Elliptic Operator of Second Order. Southwest Texas State University, San Marcos.

[12] Benilan, P., Brezis, H. and Crandall, M.G (1975) A Semilinear Equation in L^1. *Annali della Scuola Normale Superiore di Pisa-Classe di Scienze*, **2**, 523-555.

[13] Fabrie, P. and Gallouet. T (2000) Modelling Wells in Porous Media Flows. *Mathematical Models and Methods in Applied Sciences*, **10**, 673-709. https://doi.org/10.1142/S0218202500000367

[14] Kinderlehrer, D. and Pedregal, P. (1994) Gradient Young Measures Generated by Sequences in Sobolev Spaces. *The Journal of Geometric Analysis*, **4**, Article No. 59. https://doi.org/10.1007/BF02921593

[15] Browder, F.E. (1970) Existence Theorems for Nonlinear Partial Differential Equations. Global Analysis (Berkeley, 1968). In: *Proceedings of Symposia in Pure Mathematics Publication*, Vol. 16, AMS, Providence, 1-60.

https://doi.org/10.1090/pspum/016/0269962

[16] Drabek, P., Kufnen, A. and Mustonen, V. (1998) Pseudo-Monotinicity and Degenerated, a Singular Operators. *Bulletin of the Australian Mathematical Society*, **58**, 213-221. https://doi.org/10.1017/S0004972700032184

[17] Landes, R. and Mustonen, V. (1980) On Pseudo-Monotone Operators and Nonlinear Noncoercive Variational Problems on Unbounded Domains. *Mathematische Annalen*, **248**, 241-246. https://doi.org/10.1007/BF01420527

[18] Zeidler, E. (1986) Nonlinear Functional Analysis and Its Application. Vol. I, Springer, New York.