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Research Article

Spin Splitting Spectroscopy of Heavy Quark and Antiquarks Systems

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Phenomenological potentials describe the quarkonium systems like $c\bar{c}$, $b\bar{b}$, and $\bar{b}c$ where they give a good accuracy for the mass spectra. In the present work, we extend one of our previous works in the central case by adding spin-dependent terms to allow for relativistic corrections. By using such terms, we get better accuracy than previous theoretical calculations. In the present work, the mass spectra of the bound states of heavy quarks $c\bar{c}$, $b\bar{b}$, and Bc mesons are studied within the framework of the nonrelativistic Schrödinger equation. First, we solve Schrödinger's equation by Nikiforov-Uvarov (NU) method. The energy eigenvalues are presented using our new potential. The results obtained are in good agreement with the experimental data and are better than the previous theoretical estimates.

1. Introduction

In the twentieth century, quarkonium systems have been discovered. Theorists have been trying to explain some aspects of those systems like mass spectra and decay mode properties. [1–5]. Some of them used lattice quantum chromodynamics view [6–12], effective field theory [13], relativistic potential models [14, 15], semirelativistic potential models [16], and nonrelativistic potential models [17–19] which have shared in common the Coulomb and linear potentials. There are other groups which use confinement power potential r^n [20–22], the Bethe-Salpeter approach [23–25]. In the present work, we use mixed potential: nonrelativistic potential models (Coulomb+linear) and confinement power potentials plus spin-dependent splitting terms as a correction. Schrödinger's equation is solved by the Nikiforov-Uvarov (NU) method [26–34], which gives asymptotic expressions for the eigenfunctions and eigenvalues of the Schrödinger equation.

2. Methodology

In the quarkonium system which deals with quark and antiquark interaction in the center of mass frame, the masses of the quark and antiquark are bigger than chromodynamics scaling, i.e., $M_{q,\bar{q}} \gg \Lambda_{\mathrm{QCD}}$. So, this allows for nonrelativistic treatment and is considered as heavy bound systems. Using Schrödinger's equation of the two-body system in a spherical symmetric potential one obtains the following:

$$\frac{d^2 Q(r)}{dr^2} + \left[\frac{2\mu}{\hbar^2}\left(E - V_{\mathrm{tot}}(r)\right) - \frac{l(l+1)}{r^2}\right]Q(r) = 0, \quad (1)$$

where μ is reduced mass, E is energy eigenvalue, l is orbital quantum number, $V_{\mathrm{tot}}(r)$ is total potential of the system, $Q(r) = rR(r)$, and $R(r)$ is a radial wave function solution of Schrödinger's equation.Our radial potential is taken as follows:

$$V(r)=\frac{-b}{r}+ar+dr^2+pr^4. \tag{2}$$

Also, we use spin-dependent splitting terms: spin-spin interaction, spin-orbital interaction, and tensor interaction, respectively [35–43].

$$V_{s-l-T}(r)=V_{S-S}(r)+V_{s-l}(r)+V_T(r), \tag{3}$$

where

$$\begin{aligned}V_{S-S}(r)&=\frac{2}{3m_q m_{\bar q}}\nabla^2 V_V(r)\left[\overrightarrow{S_q}\bullet\overrightarrow{S_{\bar q}}\right],\\ V_{s-l}(r)&=\frac{1}{2m_q m_{\bar q} r}\left[3\frac{dV_V(r)}{dr}-\frac{dV_s(r)}{dr}\right]\left[\overrightarrow{L}\bullet\overrightarrow{S}\right],\\ V_T(r)&=\frac{1}{12m_q m_{\bar q}}\left[\frac{1}{r}\frac{dV_V(r)}{dr}-\frac{d^2V_V(r)}{dr^2}\right]\\ &\quad\cdot\left[6\left(\overrightarrow{S_q}\bullet\frac{\overrightarrow{r}}{|r|}\right)\left(\overrightarrow{S_{\bar q}}\bullet\frac{\overrightarrow{r}}{|r|}\right)-2\overrightarrow{S_q}\bullet\overrightarrow{S_{\bar q}}\right].\end{aligned} \tag{4}$$

V_V is a vector potential term, and V_s is a scalar potential term.

So, the total potential becomes

$$\begin{aligned}V_{\mathrm{tot}}(r)&=V(r)+V_{s-l-T}(r),\\ V_{\mathrm{tot}}(r)&=\frac{4a_v}{3m_q m_{\bar q} r}v(ss)+\frac{v(ls)}{m_q m_{\bar q}}\left[\frac{3a_v-a_S}{2r}+\frac{3b}{2r^3}-d-2pr^2\right]\\ &\quad+\frac{v(T)}{12m_q m_{\bar q}}\left[\frac{a_v}{r}+\frac{3b}{r^3}\right]-\frac{b}{r}+ar+dr^2+pr^4,\end{aligned} \tag{5}$$

where $v(ss)=[\overrightarrow{S_q}\bullet\overrightarrow{S_{\bar q}}]$, $v(ls)=[\overrightarrow{L}\bullet\overrightarrow{S}]$, $v(T)=-2[\overrightarrow{S_q}\bullet\overrightarrow{S_{\bar q}}-3(\overrightarrow{S_q}\bullet\overrightarrow{r}/|r|)(\overrightarrow{S_{\bar q}}\bullet\overrightarrow{r}/|r|)]a_S+a_v=a$.

By substituting in equation (1), we get

$$\frac{d^2Q}{dr^2}+\left[\varepsilon-\frac{B}{r^3}-\frac{l(l+1)}{r^2}-\frac{C}{r}-Ar-Fr^2-Pr^4\right]Q=0, \tag{6}$$

where in natural units,

$$\varepsilon=2\mu E+\frac{2\mu d v(ls)}{m_q m_{\bar q}}, \tag{7}$$

$$B=\frac{\mu b[6v(ls)+v(t)]}{2m_q m_{\bar q}}. \tag{8}$$

$$A=2\mu a,\ F=\frac{2\mu}{m_q m_{\bar q}}\left[-2pv(ls)+dm_q m_{\bar q}\right], \tag{9}$$

$$C=\frac{\mu\left[16a_v v(ss)+6(3a_v-a_S)v(ls)+a_v v(T)-12m_q m_{\bar q} b\right]}{6m_q m_{\bar q}},\ P=2\mu p. \tag{10}$$

Let $x=1/r$, and by substituting in equation (6), we obtain

$$\frac{d^2Q}{dx^2}+\frac{2}{x}\frac{dQ}{dx}+\frac{1}{x^2}\left[\frac{\varepsilon}{x^2}-Bx-\frac{C}{x}-l(l+1)-\frac{A}{x^3}-\frac{F}{x^4}-\frac{P}{x^6}\right]Q=0. \tag{11}$$

In equation (11), one can use the Nikiforov-Uvarov method (NU) to get eigenvalue and eigenfunction equations.

Due to the singularity point in equation (11), put $y+\delta=x$, and using the Taylor series to expand to second order terms, one obtains

$$\frac{d^2Q}{dx^2}+\frac{2}{x}\frac{dQ}{dx}+\frac{1}{x^2}\left[-q+wx-zx^2\right]Q=0, \tag{12}$$

where

$$-\frac{6\varepsilon}{\delta^2}+\frac{3C}{\delta}+l(l+1)+\frac{10A}{\delta^3}+\frac{15F}{\delta^4}+\frac{28P}{\delta^6}=q, \tag{13}$$

$$-\frac{8\varepsilon}{\delta^3}-B+\frac{3C}{\delta^2}+\frac{15A}{\delta^4}+\frac{24F}{\delta^5}+\frac{48P}{\delta^7}=w, \tag{14}$$

$$-\frac{3\varepsilon}{\delta^4}+\frac{C}{\delta^3}+\frac{6A}{\delta^5}+\frac{10F}{\delta^6}+\frac{21P}{\delta^8}=z. \tag{15}$$

We get

$$z=\left[\frac{w}{2n+1+2\sqrt{q+1/4}}\right]^2. \tag{16}$$

By substituting equations (13)–(15) in equation (16) and arrange it, we get

$$\begin{aligned}\varepsilon&=\frac{C\delta}{3}+\frac{2A}{\delta}+\frac{10F}{3\delta^2}+\frac{7P}{\delta^4}-\frac{1}{3}\\ &\quad\cdot\left[\frac{3C+15A/\delta^2+24F/\delta^3+48P/\delta^5-8\epsilon/8\epsilon-B\delta^2}{2n+1+2\sqrt{(l+1/2)^2+3C/\delta-6\epsilon/\delta^2+10A/\delta^3+15F/\delta^4+28P/\delta^6}}\right]^2.\end{aligned} \tag{17}$$

We substitute equations (8)–(10) into equation (17) to obtain the energy eigenvalue equation.

$$E=\aleph-\frac{\mu}{6}\left[\frac{\xi}{2n+1+2\sqrt{(l+1/2)^2+\mu\eta}}\right]^2, \tag{18}$$

where $\delta=1/r_0$, $\aleph=([16a_v v(ss)+6(3a_v-a_S)v(ls)+a_v v(t)-12m_q m_{\bar q} b]/36m_q m_{\bar q} r_0)+2ar_0+(10/3m_q m_{\bar q})[-2pv(ls)+dm_q m_{\bar q}]r_0{}^2+7pr_0{}^4-dv(ls)/m_q m_{\bar q}$,

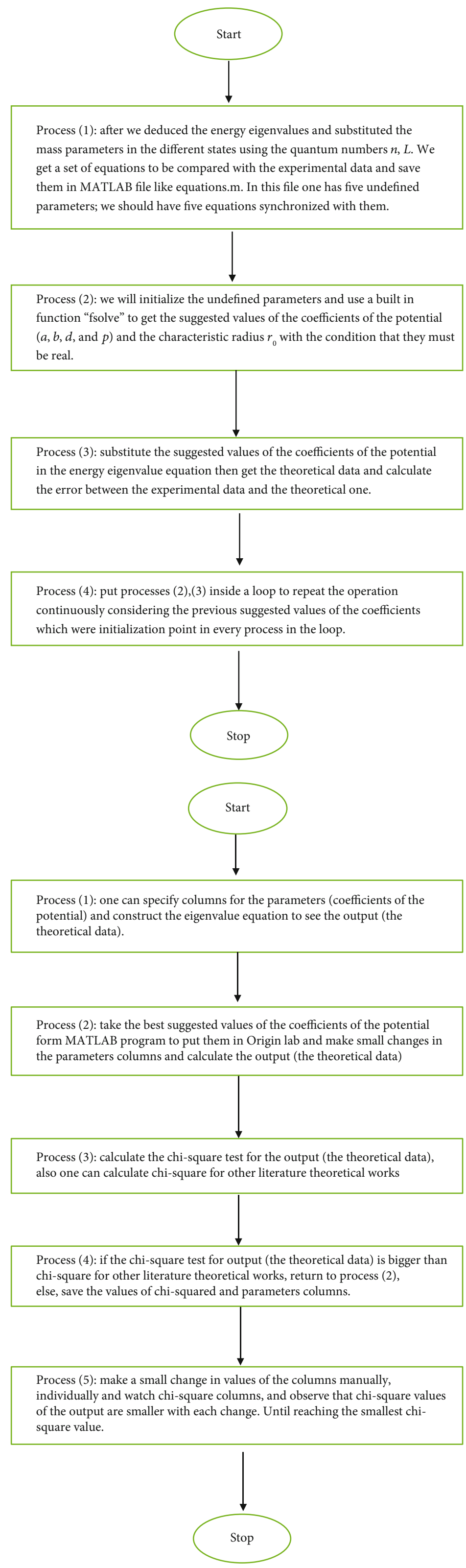

Figure 1: This is a flow chart about how one can calculate the theoretical data. It consists of three stages to do that. We found that after some iteration processes in the loops, the previous suggested values of the coefficients are the same in all last processes, as there is no change in the data. So, we used another program which gives a small change in the parameter values which was Origin lab program. In the third stage, after we got the values of the coefficients of the potential with the smallest chi-square values from the Origin program, we will substitute them in the energy eigenvalue equation and calculate the theoretical data (present work) for the different n and L states by using "fsolve" built in function in MATLAB like first flow chart.

Table 1: Parameter values of each system.

Variables Systems	m_q	$m_{\bar{q}}$	r_0	a_s	a_v	b	d	p
Units	GeV	GeV	GeV^{-1}	GeV^2	GeV^2	–	GeV^3	GeV^5
cc system	1.317	1.317	12.82	-0.0796	-0.7349	0.009	0.10686	-0.000184
bb system	4.584	4.584	7.23795	0.0505	2.5771	15.02	-0.07969	-0.00102
$\bar{b}c$ system	4.584	1.317	11.434	-2.5549	1.5369	0.039	0.11453	-0.00018438

Table 2: Charmonia mass spectrum of S and P waves in GeV.

Level	Present work	[24]	[46]	[37]	[47]	[48]	[15]	[49]	[40]	[50]	[51]	Exp. [44]
1^1S_0	3.033	2.93	2.981	2.984	2.989	2.979	2.980	2.980	2.982	3.088	2.979	2.984 (49)
1^3S_1	3.126	3.11	3.096	3.097	3.094	3.097	3.097	3.097	3.090	3.168	3.096	3.097 (29)
2^1S_0	3.666	3.68	3.635	3.637	3.602	3.623	3.597	3.633	3.630	3.669	3.600	3.639 (27)
2^3S_1	3.701	3.68	3.685	3.679	3.681	3.673	3.685	3.690	3.672	3.707	3.680	3.686 (15)
3^1S_0	4.158	–	3.989	4.004	4.058	3.991	4.014	3.992	4.043	4.067	4.011	–
3^3S_1	4.055	3.80	4.039	4.030	4.129	4.022	4.095	4.030	4.072	4.094	4.077	4.039 (16)
4^1S_0	4.415	–	4.401	4.264	4.448	4.250	4.433	4.244	4.384	4.398	4.397	–
4^3S_1	4.415	–	4.427	4.281	4.514	4.273	4.477	4.273	4.406	4.420	4.454	4.421 (6)
5^1S_0	4.607	–	4.811	4.459	4.799	4.446	–	4.440	–	–	–	–
5^3S_1	4.585	–	4.837	4.472	4.863	4.463	–	4.464	–	–	–	–
6^1S_0	4.754	–	5.155	–	5.124	4.595	–	4.601	–	–	–	–
6^3S_1	4.733	–	5.167	–	5.185	4.608	–	4.621	–	–	–	–
1^3P_0	3.407	3.32	3.413	3.415	3.428	3.433	3.416	3.392	3.424	3.448	3.488	3.415 (8)
1^3P_1	3.487	3.49	3.511	3.521	3.468	3.510	3.508	3.491	3.505	3.520	3.514	3.511 (24)
1^1P_1	3.502	3.43	3.525	3.526	3.470	3.519	3.527	3.524	3.516	3.536	3.536	3.525 (23)
1^3P_2	3.522	3.55	3.555	3.553	3.480	3.556	3.558	3.570	3.556	3.564	3.565	3.556 (33)
2^3P_0	3.899	3.83	3.870	3.848	3.897	3.842	3.844	3.845	3.852	3.870	3.947	3.918 (19)
2^3P_1	3.786	3.67	3.906	3.914	3.938	3.901	3.940	3.902	3.925	3.934	3.972	–
2^1P_1	3.821	3.75	3.926	3.916	3.943	3.908	3.960	3.922	3.934	3.950	3.996	–
2^3P_2	3.905	–	3.949	3.937	3.955	3.937	3.994	3.949	3.972	3.976	4.021	3.927 (21)
3^3P_0	4.120	–	4.301	4.146	4.296	4.131	–	4.192	4.202	4.214	–	–
3^3P_1	4.123	3.91	4.319	4.192	4.338	4.178	–	4.178	4.271	4.275	–	–
3^1P_1	4.164	–	4.337	4.193	4.344	4.184	–	4.137	4.279	4.291	–	–
3^3P_2	4.144	–	4.354	4.211	4.358	4.208	–	4.212	4.317	4.316	–	–
4^3P_0	4.362	–	4.698	–	4.653	–	–	–	–	–	–	–
4^3P_1	4.373	–	4.728	–	4.696	–	–	–	–	–	–	–
4^1P_1	4.42	–	4.744	–	4.704	–	–	–	–	–	–	–
4^3P_2	4.411	–	4.763	–	4.718	–	–	–	–	–	–	–
5^3P_0	4.543	–	–	–	4.983	–	–	–	–	–	–	–
5^3P_1	4.561	–	–	–	5.026	–	–	–	–	–	–	–
5^1P_1	4.611	–	–	–	5.034	–	–	–	–	–	–	–

Table 3: Charmonia mass spectrum of D and F waves in GeV.

Level	Present work	[47]	[46]	[37]	[24]	[48]	[15]	[49]	[40]	[50]	[51]
1^3D_3	3.307	3.755	3.813	3.808	3.869	3.799	3.831	3.844	3.806	3.809	3.798
1^1D_2	3.376	3.765	3.807	3.805	3.739	3.796	3.824	3.802	3.799	3.803	3.796
1^3D_2	3.348	3.772	3.795	3.807	3.550	3.798	3.824	3.788	3.800	3.804	3.794
1^3D_1	3.374	3.775	3.783	3.792	–	3.787	3.804	3.729	3.785	3.789	3.792
2^3D_3	3.797	4.176	4.220	4.112	3.806	4.103	4.202	4.132	4.167	4.167	4.425
2^1D_2	3.836	4.182	4.196	4.108	–	4.099	4.191	4.105	4.158	4.158	4.224
2^3D_2	3.801	4.188	4.190	4.109	–	4.100	4.189	4.095	4.158	4.159	4.223
2^3D_1	3.800	4.188	4.105	4.095	–	4.089	4.164	4.057	4.142	4.143	4.222
3^3D_3	4.163	4.549	4.574	4.340	–	4.331	–	4.351	–	–	–
3^1D_2	4.176	4.553	3.549	4.336	–	4.326	–	4.330	–	–	–
3^3D_2	4.135	4.557	4.544	4.337	–	4.327	–	4.322	–	–	–
3^3D_1	4.113	4.555	4.507	4.324	–	4.317	–	4.293	–	–	–
4^3D_3	4.436	4.890	4.920	–	–	–	–	4.526	–	–	–
4^1D_2	4.429	4.892	4.898	–	–	–	–	4.509	–	–	–
4^3D_2	4.383	4.896	4.896	–	–	–	–	4.504	–		–
4^3D_1	4.345	4.891	4.857	–	–	–	–	4.480	–	–	–
1^3F_2	3.403	3.990	4.041	–	–	–	4.068	–	4.029	–	–
1^3F_3	3.375	4.012	4.068	–	3.999	–	4.070	–	4.029	–	–
1^1F_3	3.403	4.017	4.071	–	4.037	–	4.066	–	4.026	–	–
1^3F_4	3.315	4.036	4.093	–	–	–	4.062	–	4.021	–	–
2^3F_2	3.812	4.378	4.361	–	–	–	–	–	4.351	–	–
2^3F_3	3.823	4.396	4.400	–	–	–	–	–	3.352	–	–
2^1F_3	3.858	4.400	4.406	–	–	–	–	–	4.350	–	–
2^3F_4	3.814	4.415	4.434	–	–	–	–	–	4.348	–	–
3^3F_2	4.111	4.730	–	–	–	–	–	–	–	–	–
3^3F_3	4.152	4.746	–	–	–	–	–	–	–	–	–
3^1F_3	4.194	4.749	–	–	–	–	–	–	–	–	–
3^3F_4	4.186	4.761	–	–	–	–	–	–	–	–	–

Table 4: Bottomonia mass spectrum of S and P waves in GeV.

Level	Present work	[52]	[46]	[37]	[24]	[53]	[15]	[49]	[54]	Exp. [44]
1^1S_0	9.472	9.402	9.398	9.390	9.414	9.389	9.393	9.392	9.455	9.398 (74)
1^3S_1	9.525	9.465	9.460	9.460	9.490	9.460	9.460	9.460	9.502	9.460 (65)
2^1S_0	10.028	9.976	9.990	9.990	9.987	9.987	9.987	9.991	9.990	9.999 (28)
2^3S_1	10.049	10.003	10.023	10.015	10.089	10.016	10.023	10.024	10.015	10.023 (26)
3^1S_0	10.36	10.336	10.329	10.326	–	10.330	10.345	10.323	10.330	–
3^3S_1	10.371	10.354	10.355	10.343	10.327	10.351	10.364	10.346	10.349	10.355 (16)
4^1S_0	10.592	10.523	10.573	10.584	–	10.595	10.623	10.558	–	–
4^3S_1	10.598	10.635	10.586	10.597	–	10.611	10.643	10.575	10.607	10.579 (19)
5^1S_0	10.79	10.869	10.851	10.800	–	10.817	–	10.741	–	–
5^3S_1	10.87	10.878	10.869	10.811	–	10.831	–	10.755	10.818	10.876 (6)
6^1S_0	10.961	11.097	11.061	10.997	–	11.011	–	10.892	–	–
6^3S_1	11.022	11.102	11.088	10.988	–	10.988	–	10.904	10.995	11.019 (3)
1^3P_0	9.84	9.847	9.859	9.864	9.815	9.865	9.861	9.862	9.855	9.859 (19)
1^3P_1	9.875	9.876	9.892	9.903	9.842	9.897	9.891	9.888	9.874	9.893 (18)
1^1P_1	9.884	9.882	9.900	9.909	9.806	9.903	9.900	9.896	9.879	9.899 (15)
1^3P_2	9.903	9.897	9.912	9.921	9.906	9.918	9.912	9.908	9.886	9.912 (9)
2^3P_0	10.202	10.226	10.233	10.220	10.254	10.226	10.230	10.241	10.221	10.232 (30)
2^3P_1	10.229	10.246	10.255	10.249	10.120	10.251	10.255	10.256	10.236	10.255 (26)
2^1P_1	10.237	10.250	10.260	10.254	10.154	10.256	10.262	10.261	10.240	10.260 (23)
2^3P_2	10.254	10.261	10.268	10.264	–	10.269	10.271	10.268	10.246	10.269 (15)
3^3P_0	10.299	10.552	10.521	10.490	–	10.502	–	10.511	10.500	–
3^3P_1	10.339	10.538	10.541	10.515	10.303	10.524	–	10.507	10.513	–
3^1P_1	10.362	10.541	10.544	10.519	–	10.529	–	10.497	10.516	–
3^3P_2	10.406	10.550	10.550	10.528	–	10.540	–	10.516	10.521	–
4^3P_0	10.532	10.775	10.781	–	–	10.732	–	–	–	–
4^3P_1	10.571	10.788	10.802	–	–	10.753	–	–	–	–
4^1P_1	10.594	10.790	10.804	–	–	10.757	–	–	–	–
4^3P_2	10.637	10.798	10.812	–	–	10.767	–	–	–	–
5^3P_0	10.731	11.004	–	–	–	10.933	–	–	–	–
5^3P_1	10.769	11.014	–	–	–	10.951	–	–	–	–
5^1P_1	10.792	11.016	–	–	–	10.955	–	–	–	–

Table 5: Bottomonia mass spectrum of D and F waves in GeV.

Level	Present work	[52]	[46]	[37]	[24]	[53]	[15]	[49]	[54]	Exp. [44]
1^3D_3	9.849	10.115	10.166	10.157	10.232	10.156	10.163	10.177	10.127	–
1^1D_2	9.767	10.148	10.163	10.153	10.194	10.152	10.158	10.166	10.123	–
1^3D_2	10.096	10.147	10.161	10.153	10.145	10.151	10.157	10.162	10.122	10.163 (66)
1^3D_1	9.666	10.138	10.154	10.146	–	10.145	10.149	10.147	10.117	–
2^3D_3	10.175	10.455	10.449	10.436	–	10.442	10.456	10.447	10.422	–
2^1D_2	10.093	10.450	10.445	10.432	–	10.439	10.452	10.440	10.419	–
2^3D_2	10.071	10.449	10.443	10.432	–	10.438	10.450	10.437	10.418	–
2^3D_1	9.996	10.441	10.435	10.425	–	10.432	10.443	10.428	10.414	–
3^3D_3	10.446	10.711	10.717	–	–	10.680	–	10.652	–	–
3^1D_2	10.368	10.706	10.713	–	–	10.677	–	10.646	—	–
3^3D_2	10.345	10.705	10.711	–	–	10.676	–	10.645	–	–
3^3D_1	10.272	10.698	10.704	–	–	10.670	–	10.637	–	–
4^3D_3	10.676	10.939	10.963	–	–	10.886	–	10.817	–	–
4^1D_2	10.599	10.935	10.959	–	–	10.883	–	10.813	–	–
4^3D_2	10.576	10.934	10.957	–	–	10.882	–	10.811	–	–
4^3D_1	10.504	10.928	10.949	–	–	10.877	–	10.811	–	–
1^3F_2	9.642	10.350	10.343	10.338	–	–	10.353	–	10.315	–
1^3F_3	9.754	10.355	10.346	10.340	10.302	–	10.356	–	10.321	–
1^1F_3	9.778	10.355	10.347	10.339	10.319	–	10.356	–	10.322	–
1^3F_4	9.896	10.358	10.349	10.340	–	–	10.357	–	–	–
2^3F_2	9.971	10.615	10.610	–	–	–	10.610	–	–	–
2^3F_3	10.081	10.619	10.614	–	–	–	10.613	–	–	–
2^1F_3	10.104	10.619	10.647	–	–	–	10.613	–	–	–
2^3F_4	10.219	10.622	10.617	–	–	–	10.615	–	–	–
3^3F_2	10.246	10.850	–	–	–	–	–	–	–	–
3^3F_3	10.353	10.853	–	–	–	–	–	–	–	–
3^1F_3	10.376	10.853	–	–	–	–	–	–	–	–
3^3F_4	10.489	10.856	–	–	–	–	–	–	–	–

Table 6: B_c meson mass spectrum of S and P waves in GeV.

Level	Present work	[47]	[55]	[46]	[56]	[57]	Exp.[44]
1^1S_0	6.276	6.272	6.278	6.272	6.271	6.275	6.275 (0)
1^3S_1	6.313	6.321	6.331	6.333	6.338	6.314	–
2^1S_0	6.841	6.864	6.863	6.842	6.855	6.838	6.842 (1)
2^3S_1	6.867	6.900	6.873	6.882	6.887	6.850	–
3^1S_0	7.281	7.306	7.244	7.226	7.250	–	–
3^3S_1	7.308	7.338	7.249	7.258	7.272	–	–
4^1S_0	7.634	7.684	7.564	7.585	–	–	–
4^3S_1	7.66	7.714	7.568	7.609	–	–	–
5^1S_0	7.917	8.025	7.852	7.928	–	–	–
5^3S_1	7.941	8.054	7.855	7.947	–	–	–
6^1S_0	8.144	8.340	8.120	–	–	–	–
6^3S_1	8.168	8.368	8.122	–	–	–	–
1^3P_0	6.223	6.686	6.748	6.699	6.706	6.672	–
1^3P_1	6.281	6.705	6.767	6.750	6.741	6.766	–
1^1P_1	6.290	6.706	6.769	6.743	6.750	6.828	–
1^3P_2	6.366	6.712	6.775	6.761	6.768	6.776	–
2^3P_0	6.782	7.146	7.139	7.094	7.122	6.914	–
2^3P_1	6.836	7.165	7.155	7.134	7.145	7.259	–
2^1P_1	6.846	7.168	7.156	7.094	7.150	7.322	–
2^3P_2	6.917	7.173	7.162	7.157	7.164	7.232	–
3^3P_0	7.227	7.536	7.463	7.474	–	–	–
3^3P_1	7.278	7.555	7.479	7.510	–	–	–
3^1P_1	7.287	7.559	7.479	7.500	–	–	–
3^3P_2	7.355	7.565	7.485	7.524	–	–	–
4^3P_0	7.583	7.885	–	7.817	–	–	–
4^3P_1	7.631	7.905	–	7.853	–	–	–
4^1P_1	7.640	7.908	–	7.844	–	–	–
4^3P_2	7.704	7.915	–	7.867	–	–	–
5^3P_0	7.867	8.207	–	–	–	–	–
5^3P_1	7.913	8.226	–	–	–	–	–
5^1P_1	7.922	8.230	–	–	–	–	–

Table 7: B_c meson mass spectrum of D and F waves in GeV.

Level	Present work	[47]	[55]	[46]	[56]	[57]
1^3D_3	6.429	6.990	7.026	7.029	7.045	6.980
1^1D_2	6.308	6.994	7.035	7.026	7.041	7.009
1^3D_2	6.299	6.997	7.025	7.025	7.036	7.154
1^3D_1	6.200	6.998	7.030	7.021	7.028	7.078
2^3D_3	6.975	7.399	7.363	7.405	–	–
2^1D_2	6.861	7.401	7.370	7.400	–	–
2^3D_2	6.852	7.403	7.361	7,399	–	–
2^3D_1	6.759	7.403	7.365	7.392	–	–
3^3D_3	7.409	7.761	–	7.750	–	–
3^1D_2	7.3017	7.762	–	7.743	–	–
3^3D_2	7.292	7.764	–	7.741	–	–
3^3D_1	7.205	7.762	–	7.732	–	–
4^3D_3	7.754	8.092	–	–	–	–
4^1D_2	7.653	8.093	–	–	–	–
4^3D_2	7.643	8.094	–	–	–	–
4^3D_1	7.56	8.091	–	–	–	–
1^3F_2	6.183	7.234	–	7.273	7.269	–
1^3F_3	6.326	7.242	–	7.269	7.276	–
1^1F_3	6.335	7.241	–	7.268	7.266	–
1^3F_4	6.501	7.244	–	7.277	7.271	–
2^3F_2	6.741	7.607	–	7.618	–	–
2^3F_3	6.876	7.615	–	7.616	–	–
2^1F_3	6.885	7.614	–	7.615	–	–
2^3F_4	7.041	7.617	–	7.617	–	–
3^3F_2	7.186	7.946	–	–	–	–
3^3F_3	7.314	7.954	–	–	–	–
3^1F_3	7.323	7.953	–	–	–	–
3^3F_4	7.470	7.956	–	–	–	–

$$\xi = \frac{\left[16a_v v(ss) + 6(3a_v - a_S)v(ls) + a_v v(t) - 12m_q m_{\bar{q}} b\right]}{2m_q m_{\bar{q}}}$$
$$\cdot + 30ar_0{}^2 + \frac{48}{m_q m_{\bar{q}}}\left[-2pv(ls) + dm_q m_{\bar{q}}\right] r_0{}^3$$
$$\cdot + 96pr_0{}^5 - 8r_0\left[2E + \frac{2dv(ls)}{m_q m_{\bar{q}}}\right] - \frac{b[6v(ls) + v(t)]}{2m_q m_{\bar{q}} r_0{}^2},$$

$$\eta = \left[\frac{\left[16a_v v(ss) + 6(3a_v - a_S)v(ls) + a_v v(t) - 12m_q m_{\bar{q}} b\right] r_0}{2m_q m_{\bar{q}}}\right.$$
$$\cdot + 20ar_0{}^3 + \frac{30}{m_q m_{\bar{q}}}\left[-2pv(ls) + dm_q m_{\bar{q}}\right] r_0{}^4$$
$$\left.\cdot + 56pr_0{}^6 - 6r_0{}^2\left[2E + \frac{2dv(ls)}{m_q m_{\bar{q}}}\right]\right]. \quad (19)$$

Knowing that

$$M(q\bar{q}) = E + m_q + m_{\bar{q}} \rightarrow E = M(q\bar{q}) - \left(m_q + m_{\bar{q}}\right). \quad (20)$$

So, the mass spectra equation becomes

$$M(q\bar{q}) = \aleph - \frac{\mu}{6}\left[\frac{\xi}{2n + 1 + 2\sqrt{(l + 1/2)^2 + \mu\eta}}\right]^2 + m_q + m_{\bar{q}}. \quad (21)$$

The eigenfunction equation is

$$\mathrm{Q}(\mathrm{r}) = N_{nls} r^{1/2 - \sqrt{(q+1/4)}} e^{-\sqrt{z}/r} L_n^{2\sqrt{(q+1/4)}}\left(\frac{2\sqrt{z}}{r}\right), \quad (22)$$

where $L_n^{2\sqrt{(q+1/4)}}(2\sqrt{z}/r)$ is the Rodrigues formula of the associated Laguerre polynomial and N_{nls} is a normalization constant.

So, the radial wave function solution of Schrödinger's equation is given by

$$R(r) = N_{nls} r^{-1/2 - \sqrt{(q+1/4)}} e^{-\sqrt{z}/r} L_n^{2\sqrt{(q+1/4)}}\left(\frac{2\sqrt{z}}{r}\right). \quad (23)$$

The energy eigenvalue equation (18) has spin-orbital-tensor coefficients $v(ss), v(sl), v(T)$, and those can be given from references [35–37]. Also, it has potential parameters (a_S, a_v, b, d, p) and r_0 due to the expansion, so we have six parameters of the eigenvalue equation which can be obtained from the experimental data [44] by best fitting as shown in Figure 1.

3. Numerical Results and Discussions

In Table 1, potential parameters are shown for each system. It is noticed that the values of these parameters are different for different systems, and this is due to the properties of those systems like energy scale and decay mode. We use spectroscopic notation for the levels $(n^{2S+1}L_J)$.

S is the total spin of the system, L is the orbital quantum number, n is the principal quantum number, and J is the total (orbital+spin) quantum number.

By using equation (21) and Table 1, we get the mass spectra of different quantum states as shown in Tables 2–7. Previously, we used the phenomenological potential in equation (2) without spin-dependent corrections (central-dependent potential) [45]. The results obtained were good in comparison with the experimental data.

4. Conclusions

The above tables show that spin-dependent terms are important factors to give a better accuracy and complete quantitative description of the quarkonium systems for the cases where experimental values are available. The theoretical work agrees with the experimental data. This shows also that the Nikiforov-Uvarov method is a good method to get the energy eigenvalues for the meson spectra. The results are even better than other previous works.

Data Availability

I used experimental data, and this is available for all. The link https://iopscience.iop.org/article/10.1088/1674-1137/40/10/100001/meta was mentioned in the paper reference [44].

Conflicts of Interest

The authors declare that they have no conflicts of interest.

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