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Research Article

Existence Results for a Nonlocal Coupled System of Sequential Fractional Differential Equations Involving ψ-Hilfer Fractional Derivatives

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In this article, we discuss the existence and uniqueness of solutions for a new class of coupled system of sequential fractional differential equations involving ψ-Hilfer fractional derivatives, supplemented with multipoint boundary conditions. We make use of Banach's fixed point theorem to obtain the uniqueness result and the Leray-Schauder alternative to obtain the existence result. Examples illustrating the main results are also constructed.

1. Introduction

Fractional calculus is an emerging field in applied mathematics that deals with derivatives and integrals of arbitrary orders. One of the most important advantages of fractional order models in comparison with integer order ones is that fractional integrals and derivatives are a powerful tool for the description of memory and hereditary properties of some materials. For details and applications, we refer the reader to the texts [1–6]. There are some different definitions of fractional derivatives, from the most popular of Riemann-Liouville and Caputo type fractional derivatives, to the other ones such as Hadamard fractional derivative and the Erdeyl-Kober fractional derivative. A generalization of both Riemann-Liouville and Caputo derivatives was given by Hilfer in [7], which is known as *the Hilfer fractional derivative* $D^{\alpha,\beta}x(t)$ of order α and a type $\beta \in [0, 1]$. Some properties and applications of the Hilfer derivative can be found in [8, 9] and references cited therein.

Initial value problems involving Hilfer fractional derivatives were studied by several authors (see, for example, [10–12]). Nonlocal boundary value problems for Hilfer fractional differential equation have been discussed in [13, 14]. Coupled systems for Hilfer fractional differential equations with nonlocal integral boundary conditions were studied in [15].

The fractional derivative with another function, in the Hilfer sense, called ψ-Hilfer fractional derivative, has been introduced in [16], which unifies several different fractional operators. For some recent results on existence and uniqueness of initial value problems and results on Ulam-Hyers-

Rassias stability, see [17–19] and references therein. Recently, in [20], the authors extended the results in [13] to ψ-Hilfer nonlocal implicit fractional boundary value problems. For recent results in ψ-Hilfer fractional derivative, we refer to [21–23] and references cited therein.

In [24], the authors initiated the study of existence and uniqueness of solutions for a new class of boundary value problems of sequential ψ-Hilfer-type fractional differential equations with multipoint boundary conditions of the form

$$\begin{cases} \left({}^{H}D^{\alpha,\beta;\psi} + k^{H}D^{\alpha-1,\beta;\psi}\right)x(t) = f(t, x(t)), t \in [a, b], \\ x(a) = 0, x(b) = \sum_{i=1}^{m} \lambda_i x(\theta_i), \end{cases} \tag{1}$$

where ${}^{H}D^{\alpha,\beta;\psi}$ is the ψ-Hilfer fractional derivative of order α, $1 < \alpha \le 2$ and parameter β, $0 \le \beta \le 1$, $f : [a, b] \times \mathbb{R} \longrightarrow \mathbb{R}$ is a continuous function, $0 \le a < b, k, \lambda_i \in \mathbb{R}$, $i = 1, 2, \cdots, m$ and $a < \theta_1 < \theta_2 < \cdots < \theta_m < b$. Existence and uniqueness results were proved by using classical fixed point theorems. The Banach's fixed point theorem was used to obtain the uniqueness result, while nonlinear alternative of Leray-Schauder type and Krasnoselskii's fixed point theorem are applied to obtain the existence results for the problem (1).

In this paper, we investigate the existence and uniqueness criteria for the solutions of the following nonlocal coupled system of sequential ψ-Hilfer fractional derivative of the form

$$\begin{cases} \left({}^{H}D^{\alpha,\beta;\psi} + k^{H}D^{\alpha-1,\beta;\psi}\right)x(t) = f(t, x(t), y(t)), t \in [a, b], \\ \left({}^{H}D^{p,q;\psi} + \nu^{H}D^{p-1,q;\psi}\right)y(t) = g(t, x(t), y(t)), t \in [a, b], \\ x(a) = 0, x(b) = \sum_{i=1}^{m-2} \lambda_i y(\theta_i), \\ y(a) = 0, y(b) = \sum_{j=1}^{n-2} \mu_j x(\zeta_j), \end{cases} \tag{2}$$

where ${}^{H}D^{\alpha,\beta;\psi}$, ${}^{H}D^{p,q;\psi}$ are the ψ-Hilfer fractional derivatives of orders α and p, $1 < \alpha, p \le 2$, and two parameters β, q, $0 \le \beta, q \le 1$, given constants $k, \nu, \lambda_i, \mu_j \in \mathbb{R}$, $a \ge 0$, and the points $a < \theta_1 < \theta_2 < \cdots < \theta_{m-2} < b, a < \zeta_1 < \zeta_2 < \cdots < \zeta_{n-2} < b$ and $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions.

In order to study the problem (2), we convert it into an equivalent fixed point problem and then we use Banach's fixed point theorem to prove the uniqueness of its solutions, while by applying the Leray-Schauder alternative [25], we obtain the existence result.

The remaining part of the article is structured as follows: Section 3 contains the main results for the problem (2). Examples illustrating the existence and uniqueness results are also included. We recall the related background material in Section 2, in which also we establish a lemma regarding a linear variant of the problem (2).

2. Preliminaries

Here, some notations and definitions of fractional calculus are reminded [1].

Definition 1. The Riemann-Liouville fractional integral of order $\varsigma > 0$ for a continuous function is defined by

$$I^{\varsigma} u(t) = \frac{1}{\Gamma(\varsigma)} \int_a^t (t-s)^{\varsigma-1} u(s) ds, \tag{3}$$

provided the right-hand side exists on (a, ∞).

Definition 2. The Riemann-Liouville fractional derivative of order $\varsigma > 0$ of a continuous function is defined by

$$\begin{aligned} {}^{\mathrm{RL}}D^{\varsigma} u(t) &= D^n I^{n-\varsigma} u(t) \\ &= \frac{1}{\Gamma(n-\varsigma)} \left(\frac{d}{dt}\right)^n \int_a^t (t-s)^{n-\varsigma-1} u(s) ds, n-1 < \varsigma < n, \end{aligned} \tag{4}$$

where $n = [\varsigma] + 1$ denotes the integer part of real number ς and $D = d/dt$, provided the right-hand side is point-wise defined on (a, ∞).

Definition 3. The Caputo fractional derivative of order $\varsigma > 0$ of a continuous function is defined by

$$\begin{aligned} {}^{C}D^{\varsigma} u(t) &= I^{n-\varsigma} D^n u(t) \\ &= \frac{1}{\Gamma(n-\varsigma)} \int_a^t (t-s)^{n-\varsigma-1} \left(\frac{d}{ds}\right)^n u(s) ds, n-1 < \varsigma < n, \end{aligned} \tag{5}$$

where the right-hand side is point-wise defined on (a, ∞).

Definition 4 (Hilfer fractional derivative [7, 8]). The Hilfer fractional derivative of order α and parameter β of a function (also known as the generalized Riemann-Liouville and Caputo fractional derivatives) is defined by

$${}^{H}D^{\alpha,\beta} u(t) = I^{\beta(n-\alpha)} D^n I^{(1-\beta)(n-\alpha)} u(t), \tag{6}$$

where $n - 1 < \alpha < n$, $0 \le \beta \le 1$, and $t > a$.

Remark 5. When $\beta = 0$, the Hilfer fractional derivative corresponds to the Riemann-Liouville fractional derivative

$${}^{H}D^{\alpha,0} u(t) = D^n I^{n-\alpha} u(t), \tag{7}$$

while when $\beta = 1$, the Hilfer fractional derivative corresponds to the Caputo fractional derivative

$${}^{H}D^{\alpha,1} u(t) = I^{n-\alpha} D^n u(t). \tag{8}$$

Let $\psi \in C^1([a, b], \mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a, b]$.

Definition 6 ([1]). Let $\alpha > 0$ and $g \in L^1([a,b],\mathbb{R})$. The ψ-Riemann-Liouville fractional integral of order α to a function g with respect to ψ is defined by

$$I^{\alpha;\psi} g(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \psi'(s)(\psi(t)-\psi(s))^{\alpha-1} g(s) ds. \quad (9)$$

Definition 7 ([16]). Let $n-1 < \alpha < n$, $n \in \mathbb{N}$, and $g, \psi \in C^n([a,b],\mathbb{R})$ such that ψ is increasing with $\psi'(t) \neq 0$ for all $t \in [a,b]$. The ψ-Hilfer fractional derivative ${}^H D^{\alpha,\beta;\psi}(\cdot)$ of order α to a function g and type $0 \leq \beta \leq 1$ is defined by

$${}^H D^{\alpha,\beta;\psi} g(t) = I^{\beta(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I^{(1-\beta)(n-\alpha);\psi} g(t). \quad (10)$$

Remark 8 ([1]). If $\beta = 0$, then we have ψ-Riemann-Liouville fractional derivative as

$${}^H D^{\alpha,0;\psi} g(t) := {}^{\mathrm{RL}} D^{\alpha;\psi} g(t) = \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n I^{(n-\alpha);\psi} g(t), \quad (11)$$

and if $\beta = 1$, we obtain ψ-Caputo fractional derivative by

$${}^H D^{\alpha,1;\psi} g(t) := {}^C D^{\alpha;\psi} g(t) = I^{(n-\alpha);\psi} \left(\frac{1}{\psi'(t)} \frac{d}{dt} \right)^n g(t). \quad (12)$$

Lemma 9 ([16]). *Let $\alpha, \chi > 0$ and $\delta > 0$ be constants and $\psi \in C^1([a,b],\mathbb{R})$ be an increasing function with $\psi'(t) \neq 0$ for all $t \in [a,b]$. Then, we have*

$$\begin{aligned} I^{\alpha;\psi} I^{\chi;\psi} h(t) &= I^{\alpha+\chi;\psi} h(t), \\ I^{\alpha;\psi} (\psi(t)-\psi(a))^{\delta-1} &= \frac{\Gamma(\delta)}{\Gamma(\alpha+\delta)} (\psi(t)-\psi(a))^{\alpha+\delta-1}. \end{aligned} \quad (13)$$

The following lemma contains the compositional property of Riemann-Liouville fractional integral operator with the ψ-Hilfer fractional derivative operator.

Lemma 10 ([16]). *Let $f \in L(a,b)$, $n-1 < \alpha \leq n$, $n \in \mathbb{N}$, $0 \leq \beta \leq 1, \gamma^* = \alpha + n\beta - \alpha\beta$, and $(I^{(n-\alpha)(1-\beta)} f) \in AC^k[a,b]$. Then,*

$$\left(I^{\alpha;\psi\, H} D^{\alpha,\beta;\psi} f \right)(t) = f(t) - \sum_{k=1}^{n} \frac{(\psi(t)-\psi(a))^{\gamma^*-k}}{\Gamma(\gamma^*-k+1)} f_\psi^{[n-k]} \left(I^{(1-\beta)(n-\alpha);\psi} f \right)(a), \quad (14)$$

where $f_\psi^{[n-k]} = (1/\psi'(t) d/dt)^{n-k}$.

The following lemma deals with a linear variant of the system (2).

Lemma 11. *Let $\gamma = \alpha + 2\beta - \alpha\beta$, $\delta = p + 2q - pq$, and $h, z \in C([a,b],\mathbb{R})$ be given functions. Then, the unique solution of ψ-Hilfer the fractional differential linear system*

$$\begin{cases} \left({}^H D^{\alpha,\beta;\psi} + k^H D^{\alpha-1,\beta;\psi} \right) x(t) = h(t), t \in [a,b], \\ \left({}^H D^{p,q;\psi} + \nu^H D^{p-1,q;\psi} \right) y(t) = z(t), t \in [a,b], \\ x(a) = 0, x(b) = \sum_{i=1}^{m-2} \lambda_i y(\theta_i), \\ y(a) = 0, y(b) = \sum_{j=1}^{n-2} \mu_j x(\zeta_j), \end{cases} \quad (15)$$

is given by

$$\begin{aligned} x(t) = {} & I^{\alpha;\psi} h(t) - k I^{1;\psi} x(t) + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \\ & \cdot \left\{ \Delta \left[-\nu \sum_{i=1}^{m-2} \lambda_i I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I^{p;\psi} z(\theta_i) + k I^{1;\psi} x(b) - I^{\alpha;\psi} h(b) \right] \right. \\ & \left. + B \left[-k \sum_{j=1}^{n-2} \mu_j I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} \mu_j I^{\alpha;\psi} h(\zeta_j) + \nu I^{1;\psi} y(b) - I^{p;\psi} z(b) \right] \right\}, \end{aligned} \quad (16)$$

$$\begin{aligned} y(t) = {} & I^{p;\psi} z(t) - \nu I^{1;\psi} y(t) + \frac{(\psi(t)-\psi(a))^{\delta-1}}{\Lambda \Gamma(\gamma)} \\ & \cdot \left\{ A \left[-k \sum_{j=1}^{n-2} \mu_j I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} \mu_j I^{\alpha;\psi} h(\zeta_j) + \nu I^{1;\psi} y(b) - I^{p;\psi} z(b) \right] \right. \\ & \left. + \Omega \left[-\nu \sum_{i=1}^{m-2} \lambda_i I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I^{p;\psi} z(\theta_i) + k I^{1;\psi} x(b) - I^{\alpha;\psi} h(b) \right] \right\}, \end{aligned} \quad (17)$$

where

$$\begin{aligned} A &= \frac{(\psi(b)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}, \\ B &= \sum_{i=1}^{m-2} \lambda_i \frac{(\psi(\theta_i)-\psi(a))^{\delta-1}}{\Gamma(\delta)}, \\ \Omega &= \sum_{j=1}^{n-2} \mu_j \frac{(\psi(\zeta_j)-\psi(a))^{\gamma-1}}{\Gamma(\gamma)}, \\ \Delta &= \frac{(\psi(b)-\psi(a))^{\delta-1}}{\Gamma(\delta)}, \end{aligned} \quad (18)$$

and it is assumed that

$$\Lambda := A\Delta - B\Omega \neq 0. \quad (19)$$

Proof. Assume that x is a solution of the nonlocal boundary value problem (15) on $[a,b]$. Operating fractional integral $I^{\alpha;\psi}$ on both sides of the first equation in (15) and using

Lemma 10, we obtain for $t \in [a, b]$,

$$x(t) - \sum_{k=1}^{2} \frac{(\psi(t) - \psi(a))^{\gamma-k}}{\Gamma(\gamma - k + 1)} f_{\psi}^{[2-k]} \left(I^{(1-\beta)(2-\alpha);\psi} x \right)(a) + k I^{1;\psi} x(t) = I^{\alpha;\psi} h(t). \tag{20}$$

Hence, using the fact that $(1-\beta)(2-\alpha) = 2-\gamma$, we have

$$\begin{aligned} x(t) &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \left(\frac{1}{\psi'(t)} \frac{d}{dt} I^{2-\gamma;\psi} x \right)(a) \\ &\quad + \frac{(\psi(t) - \psi(a))^{\gamma-2}}{\Gamma(\gamma - 1)} I^{2-\gamma;\psi} x(a) - k I^{1;\psi} x(t) + I^{\alpha;\psi} h(t) \\ &= \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} {}^{H}D^{\gamma-1,\beta;\psi} x(a) + \frac{(\psi(t) - \psi(a))^{\gamma-2}}{\Gamma(\gamma - 1)} I^{2-\gamma;\psi} x(a) \\ &\quad - k I^{1;\psi} x(t) + I^{\alpha;\psi} h(t) = c_1 \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} \\ &\quad + c_2 \frac{(\psi(t) - \psi(a))^{\gamma-2}}{\Gamma(\gamma - 1)} - k I^{1;\psi} x(t) + I^{\alpha;\psi} h(t), \end{aligned} \tag{21}$$

where $c_1 = {}^{H}D^{\gamma-1,\beta;\psi} x(t)|_{t=a}$ and $c_2 = I^{2-\gamma;\psi} x(t)|_{t=a}$.

From the first boundary condition $x(a) = 0$, we can obtain $c_2 = 0$, since $\lim_{t \longrightarrow a} (t-a)^{\gamma-2} = \infty$. Then, we get

$$x(t) = c_1 \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Gamma(\gamma)} - k I^{1;\psi} x(t) + I^{\alpha;\psi} h(t), t \in [a, b]. \tag{22}$$

By a similar way, we obtain

$$y(t) = d_1 \frac{(\psi(t) - \psi(a))^{\delta-1}}{\Gamma(\delta)} - \nu I^{1;\psi} y(t) + I^{p;\psi} z(t), t \in [a, b], \tag{23}$$

where d_1 is an arbitrary constant.

From the second boundary conditions $x(b) = \sum_{i=1}^{m-2} \lambda_i y(\theta_i)$ and $y(b) = \sum_{j=1}^{n-2} \mu_j x(\zeta_j)$, we get the system

$$\begin{aligned} A c_1 - B d_1 &= P, \\ -\Omega c_1 + \Delta d_1 &= Q, \end{aligned} \tag{24}$$

where

$$\begin{aligned} P &= -\nu \sum_{i=1}^{m-2} \lambda_i I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I^{p;\psi} z(\theta_i) + k I^{1;\psi} x(b) - I^{\alpha;\psi} h(b), \\ Q &= -k \sum_{j=1}^{n-2} \mu_j I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} \mu_j I^{\alpha;\psi} h(\zeta_j) + \nu I^{1;\psi} y(b) - I^{p;\psi} z(b). \end{aligned} \tag{25}$$

Solving the system (24), we find that

$$c_1 = \frac{1}{\Lambda} (\Delta P + B Q), d_1 = \frac{1}{\Lambda} (A Q + \Omega P). \tag{26}$$

Substituting the value of c_1, d_1 in (22) and (23) yields the solution (16) and (17). The converse follows by direct computation. This completes the proof.

3. Main Results

Let us introduce the space $\mathscr{W} = \{x(t) \mid x(t) \in C([a, b], \mathbb{R})\}$ endowed with the norm $\|x\| = \sup \{|x(t)|, t \in [a, b]\}$. Obviously, $(\mathscr{W}, \|\cdot\|)$ is a Banach space. Then, the product space $(\mathscr{W} \times \mathscr{W}, \|(x, y)\|)$ is also a Banach space equipped with norm $\|(x, y)\| = \|x\| + \|y\|$.

In view of Lemma 11, we define an operator $\mathscr{S} : \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{W} \times \mathscr{W}$ by

$$\mathscr{S}(x, y)(t) = \begin{pmatrix} \mathscr{S}_1(x, y)(t) \\ \mathscr{S}_2(x, y)(t) \end{pmatrix}, \tag{27}$$

where

$$\begin{aligned} \mathscr{S}_1(x, y)(t) &= I^{\alpha;\psi} f_{xy}(t) - k I^{1;\psi} x(t) + \frac{(\psi(t) - \psi(a))^{\gamma-1}}{\Lambda \Gamma(\gamma)} \\ &\quad \cdot \left\{ \Delta \left[-\nu \sum_{i=1}^{m-2} \lambda_i I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I^{p;\psi} g_{xy}(\theta_i) + k I^{1;\psi} x(b) - I^{\alpha;\psi} f_{xy}(b) \right] \right. \\ &\quad \left. + B \left[-k \sum_{j=1}^{n-2} \mu_j I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} \mu_j I^{\alpha;\psi} f_{xy}(\zeta_j) + \nu I^{1;\psi} y(b) - I^{p;\psi} g_{xy}(b) \right] \right\}, \\ \mathscr{S}_2(x, y)(t) &= I^{p;\psi} g_{xy}(t) - \nu I^{1;\psi} y(t) + \frac{(\psi(t) - \psi(a))^{\delta-1}}{\Lambda \Gamma(\gamma)} \\ &\quad \cdot \left\{ A \left[-k \sum_{j=1}^{n-2} \mu_j I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} \mu_j I^{\alpha;\psi} f_{xy}(\zeta_j) + \nu I^{1;\psi} y(b) - I^{p;\psi} g_{xy}(b) \right] \right. \\ &\quad \left. + \Omega \left[-\nu \sum_{i=1}^{m-2} \lambda_i I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} \lambda_i I^{p;\psi} g_{xy}(\theta_i) + k I^{1;\psi} x(b) - I^{\alpha;\psi} f_{xy}(b) \right] \right\}, \end{aligned} \tag{28}$$

where

$$\begin{aligned} f_{xy}(t) &= f(t, x(t), y(t)), \\ g_{xy}(t) &= g(t, x(t), y(t)), t \in [a, b]. \end{aligned} \tag{29}$$

For the sake of computational convenience, we put

$$X_1 = |k|(b-a) + \frac{|A|}{|\Lambda|} |\Delta||k|(b-a) + \frac{|A|}{|\Lambda|} |B||k| \sum_{j=1}^{n-2} |\mu_j|(\zeta_j - a), \tag{30}$$

$$Y_1 = \frac{|A|}{|\Lambda|} |\Delta||\nu| \sum_{i=1}^{m-2} |\lambda_i|(\theta_i - a) + \frac{|A|}{|\Lambda|} |B||\nu|(b-a), \tag{31}$$

$$F_1 = \frac{(\psi(b) - \psi(a))^{\alpha}}{\Gamma(\alpha + 1)} \left(1 + \frac{|A|}{|\Lambda|} |\Delta| \right) + \frac{|A|}{|\Lambda|} |B| \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\zeta_j) - \psi(a))^{\alpha}}{\Gamma(\alpha + 1)}, \tag{32}$$

$$G_1 = \frac{|A|}{|\Lambda|}|\Delta|\sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)} + \frac{|A|}{|\Lambda|}|B|\frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}, \tag{33}$$

$$X_2 = \frac{|\Delta|}{|\Lambda|}|A||k|\sum_{j=1}^{n-2}|\mu_j|(\zeta_j - a) + \frac{|\Delta|}{|\Lambda|}|\Omega||k|(b-a), \tag{34}$$

$$Y_2 = |\nu|(b-a) + \frac{|\Delta|}{|\Lambda|}|A||\nu|(b-a) + \frac{|\Delta|}{|\Lambda|}|\Omega||\nu|\sum_{i=1}^{m-2}|\lambda_i|(\theta_i - a), \tag{35}$$

$$F_2 = \frac{|\Delta|}{|\Lambda|}|A|\sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\zeta_j)-\psi(a))^\alpha}{\Gamma(\alpha+1)} + \frac{|\Delta|}{|\Lambda|}|\Omega|\frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}, \tag{36}$$

$$G_2 = \frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\left(1 + \frac{|\Delta|}{|\Lambda|}|A|\right) + \frac{|\Delta|}{|\Lambda|}|\Omega|\sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)}. \tag{37}$$

Our first result is based on Leray-Schauder alternative ([25] p. 4.).

Lemma 12 (Leray-Schauder alternative). *Let $F : E \longrightarrow E$ be a completely continuous operator (i.e., a map that restricted to any bounded set in E is compact). Let*

$$\mathscr{E}(F) = \{x \in E : x = \lambda F(x) \text{ for some } 0 < \lambda < 1\}. \tag{38}$$

Then, either the set $\mathscr{E}(F)$ is unbounded, or F has at least one fixed point.

Theorem 13. *Assume that $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, and there exist real constants $p_i, q_i \geq 0$, $(i = 1, 2)$ and $p_0, q_0 > 0$ such that $\forall x_i, y_i \in \mathbb{R}$, $(i = 1, 2)$,*

$$|f(t, x_1, y_1)| \leq p_0 + p_1|x_1| + p_2|y_1|, |g(t, x_2, y_2)| \leq q_0 + q_1|x_2| + q_2|y_2|. \tag{39}$$

If

$$\begin{aligned} \mathscr{M}_1 &= [F_1 + F_2]p_1 + [G_1 + G_2]q_1 + [X_1 + X_2] < 1, \\ \mathscr{M}_2 &= [F_1 + F_2]p_2 + [G_1 + G_2]q_2 + [Y_1 + Y_2] < 1, \end{aligned} \tag{40}$$

where $X_i, Y_i, F_i, G_i, i = 1, 2$ are given by (30)-(37); then, the system (2) has at least one solution on $[a, b]$.

Proof. The operator $\mathscr{S}$ is continuous, by the continuity of functions f and g. We will show that the operator $\mathscr{S} : \mathscr{W} \times \mathscr{W} \longrightarrow \mathscr{W} \times \mathscr{W}$ is completely continuous. Let $\mathscr{Z}_r = \{(x, y) \in \mathscr{W} \times \mathscr{W} : \|(x, y)\| \leq r\}$ be bounded set. Then, there exist positive constants $\mathscr{L}_i, i = 1, 2$ such that $|f(t, x(t), y(t))| \leq \mathscr{L}_1, |g(t, x(t), y(t))| \leq \mathscr{L}_2, \forall (x, y) \in \mathscr{Z}_r$. Then, for any $(x, y) \in \mathscr{Z}_r$, we have

$$\begin{aligned} |\mathscr{S}_1(x,y)(t)| &\leq I^{\alpha;\psi}|f_{xy}(t)| + |k|I^{1;\psi}|x(t)| + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{|\Lambda|\Gamma(\gamma)} \\ &\quad \cdot \Bigg\{|\Delta|\Bigg[|\nu|\sum_{i=1}^{m-2}|\lambda_i|I^{1;\psi}y(\theta_i) \\ &\quad + \sum_{i=1}^{m-2}|\lambda_i|I^{p;\psi}|g_{xy}(\theta_i)| + |k|I^{1;\psi}|x(b)| + I^{\alpha;\psi}|f_{xy}(b)|\Bigg] \\ &\quad + |B|\Bigg[|k|\sum_{j=1}^{n-2}|\mu_j|I^{1;\psi}x(\zeta_j) \\ &\quad + \sum_{j=1}^{n-2}|\mu_j|I^{\alpha;\psi}|f_{xy}(\zeta_i)| + |\nu|I^{1;\psi}|y(b)| + I^{p;\psi}|g_{xy}(b)|\Bigg]\Bigg\} \\ &\leq \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\mathscr{L}_1 + |k|(b-a)\|x\| + \frac{|A|}{|\Lambda|} \\ &\quad \cdot \Bigg\{|\Delta|\Bigg[|\nu|\sum_{i=1}^{m-2}|\lambda_i|(\theta_i - a)\|y\| + \sum_{i=1}^{m-2}|\lambda_i|\frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)}\mathscr{L}_2 \\ &\quad + |k|(b-a)\|x\| + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\mathscr{L}_1\Bigg] + |B|\Bigg[|k|\sum_{j=1}^{n-2}|\mu_j|(\zeta_j - a)\|x\| \\ &\quad + \sum_{j=1}^{n-2}|\mu_j|\frac{(\psi(\zeta_j)-\psi(a))^\alpha}{\Gamma(\alpha+1)}\mathscr{L}_1 + |\nu|(b-a)\|y\| + \frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)}\mathscr{L}_2\Bigg]\Bigg\} \\ &\leq F_1\mathscr{L}_1 + G_1\mathscr{L}_2 + X_1\|x\| + Y_1\|y\|, \end{aligned} \tag{41}$$

which implies that

$$\|\mathscr{S}_1(x, y)\| \leq F_1\mathscr{L}_1 + G_1\mathscr{L}_2 + X_1\|x\| + Y_1\|y\|. \tag{42}$$

Similarly, it can be shown that

$$\|\mathscr{S}_2(x, y)\| \leq F_2\mathscr{L}_1 + G_2\mathscr{L}_2 + X_2\|x\| + Y_2\|y\|. \tag{43}$$

From the above inequalities, it follows that the operator $\mathscr{S}$ is uniformly bounded, since

$$\|\mathscr{S}(x, y)\| \leq [F_1 + F_2]\mathscr{L}_1 + [F_1 + G_2]\mathscr{L}_2 + [X_1 + X_2]r + [Y_1 + Y_2]r. \tag{44}$$

Next, we show that $\mathscr{S}$ is equicontinuous. Let $t_1, t_2 \in [a, b]$ with $t_1 < t_2$. Then, we have

$$\begin{aligned} &|\mathscr{S}_1(x(t_2), y(t_2)) - \mathscr{S}_1(x(t_1), y(t_1))| \\ &\quad \leq \frac{1}{\Gamma(\alpha)}\left|\int_a^{t_1}\psi'(s)\left[(\psi(t_2)-\psi(s))^{\alpha-1} - (\psi(t_1)-\psi(s))^{\alpha-1}\right]f\right. \\ &\qquad \times (s, x(s), y(s))ds + \int_{t_1}^{t_2}\psi'(s)(\psi(t_2)-\psi(s))^{\alpha-1}f \\ &\qquad \times (s, x(s), y(s))ds\Big| + |k|\int_{t_1}^{t_2}|x(s)|ds \\ &\qquad + \frac{|(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}|}{|\Lambda|\Gamma(\gamma)}|\Delta P + BQ| \\ &\quad \leq \mathscr{L}_1\frac{2(\psi(t_2)-\psi(t_1))^\alpha + |\psi(t_2)^\alpha - \psi(t_1)^\alpha|}{\Gamma(\alpha+1)} + |k|r|t_2 - t_1| \\ &\quad + \frac{|(\psi(t_2)-\psi(a))^{\gamma-1} - (\psi(t_1)-\psi(a))^{\gamma-1}|}{|\Lambda|\Gamma(\gamma)}|\Delta P + BQ|. \end{aligned} \tag{45}$$

Analogously, we can obtain

$$\begin{aligned}
&|\mathcal{S}_2(x(t_2), y(t_2)) - \mathcal{S}_2(x(t_1), y(t_1))| \\
&\quad \le \mathcal{L}_2 \frac{2(\psi(t_2)-\psi(t_1))^p + |\psi(t_2)^p - \psi(t_1)^p|}{\Gamma(p+1)} + |\nu| r |t_2 - t_1| \\
&\qquad + \frac{|(\psi(t_2)-\psi(a))^{\delta-1} - (\psi(t_1)-\psi(a))^{\delta-1}|}{|\Lambda| \Gamma(\delta)} |AQ + \Gamma P|.
\end{aligned} \tag{46}$$

Therefore, the operator $\mathcal{S}(x, y)$ is equicontinuous, and thus, the operator $\mathcal{S}(x, y)$ is completely continuous.

Finally, it will be verified that the set $\mathscr{E} = \{(x, y) \in \mathscr{W} \times \mathscr{W} \mid (x, y) = \lambda \mathcal{S}(x, y), 0 \le \lambda \le 1\}$ is bounded. Let $(x, y) \in \mathscr{E}$ with $(x, y) = \lambda \mathcal{S}(x, y)$. For any $t \in [a, b]$, we have

$$x(t) = \lambda \mathcal{S}_1(x, y)(t), y(t) = \lambda \mathcal{S}_2(x, y)(t). \tag{47}$$

Then,

$$\begin{aligned}
|x(t)| \le{}& \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} (p_0 + p_1|x| + p_2|y|) + |k|(b-a)\|x\| + \frac{|A|}{|\Lambda|} \\
&\cdot \Bigg\{ |\Delta| \Bigg[|\nu| \sum_{i=1}^{m-2} |\lambda_i| (\theta_i - a)\|y\| + \sum_{i=1}^{m-2} |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^p}{\Gamma(p+1)} \\
&\cdot (q_0 + q_1|x| + q_2|y|) + |k|(b-a)\|x\| + \frac{(\psi(b)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \\
&\cdot (p_0 + p_1|x| + p_2|y|)\Bigg] + |B| \\
&\cdot \Bigg[|k| \sum_{j=1}^{n-2} |\mu_j| (\zeta_j - a)\|x\| + \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\zeta_j)-\psi(a))^\alpha}{\Gamma(\alpha+1)} \\
&\cdot (p_0 + p_1|x| + p_2|y|) + |\nu|(b-a)\|y\| + \frac{(\psi(b)-\psi(a))^p}{\Gamma(p+1)} \\
&\cdot (q_0 + q_1|x| + q_2|y|)\Bigg]\Bigg\} \\
\le{}& F_1(p_0 + p_1|x| + p_2|y|) + G_1(q_0 + q_1|x| + q_2|y|) + X_1\|x\| + Y_1\|y\| \\
={}& (F_1 p_0 + G_1 q_0) + (p_1 F_1 + q_1 G_1 + X_1)\|x\| + (p_2 F_1 + q_2 G_1 + Y_1)\|y\|, \\
|y(t)| \le{}& F_2(p_0 + p_1|x| + p_2|y|) + G_2(q_0 + q_1|x| + q_2|y|) + X_2\|x\| + Y_2\|y\| \\
={}& (F_2 p_0 + G_2 q_0) + (p_1 F_2 + q_1 G_2 + X_2)\|x\| + (p_2 F_2 + q_2 G_2 + Y_2)\|y\|.
\end{aligned} \tag{48}$$

Hence, we have

$$\begin{aligned}
\|x\| &\le (F_1 p_0 + G_1 q_0) + (p_1 F_1 + q_1 G_1 + X_1)\|x\| + (p_2 F_1 + q_2 G_1 + Y_1)\|y\|, \\
\|y\| &\le (F_2 p_0 + G_2 q_0) + (p_1 F_2 + q_1 G_2 + X_2)\|x\| + (p_2 F_2 + q_2 G_2 + Y_2)\|y\|,
\end{aligned} \tag{49}$$

which imply that

$$\begin{aligned}
\|x\| + \|y\| \le{}& [F_1 + F_2]p_0 + [G_1 + G_2]q_0 + \{[F_1 + F_2]p_1 + [G_1 + G_2]q_1 \\
&+ [X_1 + X_2]\}\|x\| + \{[F_1 + F_2]p_2 + [G_1 + G_2]q_2 + [Y_1 + Y_2]\}\|y\|.
\end{aligned} \tag{50}$$

Consequently,

$$\|(x, y)\| \le \frac{[F_1 + F_2]p_0 + [G_1 + G_2]q_0}{\min\{1 - \mathscr{M}_1, 1 - \mathscr{M}_2\}}, \tag{51}$$

which proves that $\mathscr{E}$ is bounded. Thus, the operator $\mathcal{S}$, by Lemma 12, has at least one fixed point. Hence, the boundary value problem (2) has at least one solution. The proof is complete.

The uniqueness of solutions of the system (2) is proved in the next theorem, via Banach's contraction mapping principle.

Theorem 14. *Assume that $f, g : [a, b] \times \mathbb{R} \times \mathbb{R} \longrightarrow \mathbb{R}$ are continuous functions, and there exist positive constants $\mathscr{P}$, $\mathscr{Q}$ such that for all $t \in [a, b]$ and $u_i, v_i \in \mathbb{R}$, $i = 1, 2$, we have*

$$\begin{aligned}
|f(t, u_1, u_2) - f(t, v_1, v_2)| &\le \mathscr{P}(|u_1 - v_1| + |u_2 - v_2|), \\
|g(t, u_1, u_2) - g(t, v_1, v_2)| &\le \mathscr{Q}(|u_1 - v_1| + |u_2 - v_2|).
\end{aligned} \tag{52}$$

Then, the system (2) has a unique solution on $[a, b]$, provided that

$$[F_1 + F_2]\mathscr{P} + [G_1 + G_2]\mathscr{Q} + [X_1 + X_2] + [Y_1 + Y_2] < 1, \tag{53}$$

where $X_i, Y_i, F_i, G_i, i = 1, 2$ are given by (30)-(37).

Proof. Define $\sup_{t\in[a,b]} f(t, 0, 0) = \mathscr{N}_1 < \infty$, $\sup_{t\in[a,b]} g(t, 0, 0) = \mathscr{N}_2 < \infty$ and $r > 0$ such that

$$r > \frac{[F_1 + F_2]\mathscr{N}_1 + [G_1 + G_2]\mathscr{N}_2}{1 - \{[F_1 + F_2]\mathscr{P} + [G_1 + G_2]\mathscr{Q} + [X_1 + X_2] + [Y_1 + Y_2]\}}. \tag{54}$$

In the first step, we show that $\mathcal{S}B_r \subset B_r$, where $B_r = \{(x, y) \in \mathscr{W} \times \mathscr{W} : \|(x, y)\| \le r\}$. By the assumption (H_2), for $(x, y) \in B_r$, $t \in [a, b]$, we have

$$\begin{aligned}
|f(t, x(t), y(t))| &\le |f(t, x(t), y(t)) - f(t, 0, 0)| + |f(t, 0, 0)| \\
&\le \mathscr{P}(|x(t)| + |y(t)|) + \mathscr{N}_1, \\
&\le \mathscr{P}(\|x\| + \|y\|) + \mathscr{N}_1 \le \mathscr{P}r + \mathscr{N}_1, \\
|g(t, x(t), y(t))| &\le \mathscr{Q}r + \mathscr{N}_2.
\end{aligned} \tag{55}$$

Using the above estimates, we obtain

$$
\begin{aligned}
|\mathcal{S}_1(x,y)(t)| \le\ & I^{\alpha;\psi}|f_{xy}(t)| + |k| I^{1;\psi}|x(t)| + \frac{(\psi(t)-\psi(a))^{\gamma-1}}{|\Lambda \,|\, \Gamma(\gamma)} \\
& \cdot \Bigg\{ |\Delta| \Bigg[|\nu| \sum_{i=1}^{m-2} |\lambda_i| I^{1;\psi} y(\theta_i) + \sum_{i=1}^{m-2} |\lambda_i| I^{p;\psi} |g_{xy}(\theta_i)| \\
& + |k| I^{1;\psi} |x(b)| + I^{\alpha;\psi} |f_{xy}(b)| \Bigg] + |B| \\
& \cdot \Bigg[|k| \sum_{j=1}^{n-2} |\mu_j| I^{1;\psi} x(\zeta_j) + \sum_{j=1}^{n-2} |\mu_j| I^{\alpha;\psi} |f_{xy}(\zeta_j)| \\
& + |\nu| I^{1;\psi} |y(b)| + I^{p;\psi} |g(b,x(b),y(b))| \Bigg] \Bigg\} \\
\le\ & \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} (\mathcal{P} r + \mathcal{N}_1) + |k|(b-a)\|x\| + \frac{|A|}{|\Lambda|} \\
& \cdot \Bigg\{ |\Delta| \Bigg[|\nu| \sum_{i=1}^{m-2} |\lambda_i| (\theta_i - a)\|y\| \\
& + \sum_{i=1}^{m-2} |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^{p}}{\Gamma(p+1)} (\mathcal{Q} r + \mathcal{N}_2) + |k|(b-a)\|x\| \\
& + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} (\mathcal{P} r + \mathcal{N}_1) \Bigg] + |B| \\
& \cdot \Bigg[|k| \sum_{j=1}^{n-2} |\mu_j| (\zeta_j - a)\|x\| + \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\zeta_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \\
& \cdot (\mathcal{P} r + \mathcal{N}_1) + |\nu|(b-a)\|y\| + \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)} (\mathcal{Q} r + \mathcal{N}_2) \Bigg] \Bigg\} \\
\le\ & [F_1 \mathcal{P} + G_1 \mathcal{Q} + X_1 + Y_1] r + F_1 \mathcal{N}_1 + G_1 \mathcal{N}_2.
\end{aligned} \tag{56}
$$

Hence,

$$
\|\mathcal{S}_1(x,y)\| \le [F_1 \mathcal{P} + G_1 \mathcal{Q} + X_1 + Y_1] r + F_1 \mathcal{N}_1 + G_1 \mathcal{N}_2. \tag{57}
$$

In the same way, we can obtain that

$$
\|\mathcal{S}_2(x,y)\| \le [F_2 \mathcal{P} + G_2 \mathcal{Q} + X_2 + Y_2] r + F_2 \mathcal{N}_1 + G_2 \mathcal{N}_2. \tag{58}
$$

In consequence, it follows that

$$
\begin{aligned}
\|\mathcal{S}(x,y)\| \le\ & \{[F_1+F_2]\mathcal{P} + [G_1+G_2]\mathcal{Q} + [X_1+X_2] + [Y_1+Y_2]\} r \\
& + [F_1+F_2]\mathcal{N}_1 + [G_1+G_2]\mathcal{N}_2 \le r,
\end{aligned} \tag{59}
$$

which shows that $\mathcal{S} B_r \subset B_r$.

We prove that the operator $\mathcal{S}$ is a contraction. For $(x_2, y_2), (x_1, y_1) \in \mathcal{W} \times \mathcal{W}$ and for any $t \in [a,b]$, we get

$$
\begin{aligned}
& |\mathcal{S}_1(x_2,y_2)(t) - \mathcal{S}_1(x_1,y_1)(t)| \\
& \le \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \mathcal{P}(\|x_2-x_1\| + \|y_2-y_1\|) + |k|(b-a)\|x_2 \\
& - x_1\| + \frac{|A|}{|\Lambda|} \Bigg\{ |\Delta| \Bigg[|\nu| \sum_{i=1}^{m-2} |\lambda_i| (\theta_i - a)\|y_2 - y_1\| \\
& + \sum_{i=1}^{m-2} |\lambda_i| \frac{(\psi(\theta_i)-\psi(a))^{p}}{\Gamma(p+1)} \mathcal{Q}(\|x_2-x_1\| + \|y_2-y_1\|) \\
& + |k|(b-a)\|x_2-x_1\| + \frac{(\psi(b)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \mathcal{P}(\|x_2-x_1\| + \|y_2-y_1\|) \Bigg] \\
& + |B| \Bigg[|k| \sum_{j=1}^{n-2} |\mu_j| (\zeta_j - a)\|x\| \\
& + \sum_{j=1}^{n-2} |\mu_j| \frac{(\psi(\zeta_j)-\psi(a))^{\alpha}}{\Gamma(\alpha+1)} \mathcal{P}(\|x_2-x_1\| + \|y_2-y_1\|) \\
& + |\nu|(b-a)\|y_2-y_1\| + \frac{(\psi(b)-\psi(a))^{p}}{\Gamma(p+1)} \mathcal{Q}(\|x_2-x_1\| + \|y_2-y_1\|) \Bigg] \Bigg\} \\
& \le [F_1 \mathcal{P} + G_1 \mathcal{Q} + X_1 + Y_1](\|x_2-x_1\| + \|y_2-y_1\|),
\end{aligned} \tag{60}
$$

and consequently, we obtain

$$
\|\mathcal{S}_1(x_2,y_2) - \mathcal{S}_1(x_1,y_1)\| \le [F_1 \mathcal{P} + G_1 \mathcal{Q} + X_1 + Y_1](\|x_2-x_1\| + \|y_2-y_1\|). \tag{61}
$$

Similarly, we have

$$
\begin{aligned}
& \|\mathcal{S}_2(x_2,y_2)(t) - \mathcal{S}_2(x_1,y_1) \\
& \quad \le [F_2 \mathcal{P} + G_2 \mathcal{Q} + X_2 + Y_2](\|x_2-x_1\| + \|y_2-y_1\|).
\end{aligned} \tag{62}
$$

It follows from above two equations (61) and (62) that

$$
\begin{aligned}
\|\mathcal{S}(x_2,y_2) - \mathcal{S}(x_1,y_1)\| \le\ & \{[F_1+F_2]\mathcal{P} + [G_1+G_2]\mathcal{Q} + [X_1+X_2] + [Y_1+Y_2]\} \\
& \cdot (\|x_2-x_1\| + \|y_2-y_1\|),
\end{aligned} \tag{63}
$$

which implies that the operator $\mathcal{S}$ is a contraction, by assumption (53). Consequently, the operator $\mathcal{S}$ has a unique fixed point, by Banach's fixed point theorem, which is the unique solution of problem (2). This completes the proof.

Example 15. Consider the following system

$$
\begin{cases}
\left({}^{H}D^{\frac{3}{2},\frac{1}{3};(t^2+1)} + \dfrac{1}{55}\, {}^{H}D^{\frac{1}{2},\frac{1}{3};(t^2+1)} \right) x(t) = f(t, x(t), y(t)), \ t \in \left[\dfrac{1}{4}, \dfrac{5}{2}\right], \\
\left({}^{H}D^{\frac{4}{3},\frac{1}{2};(t^2+1)} + \dfrac{1}{58}\, {}^{H}D^{\frac{1}{3},\frac{1}{2};(t^2+1)} \right) y(t) = g(t, x(t), y(t)), \ t \in \left[\dfrac{1}{4}, \dfrac{5}{2}\right], \\
x\left(\dfrac{1}{4}\right) = 0, x\left(\dfrac{5}{2}\right) = \dfrac{1}{3} y\left(\dfrac{3}{4}\right) + \dfrac{1}{6} y\left(\dfrac{3}{2}\right) + \dfrac{1}{9} y(2), \\
\left(\dfrac{1}{4}\right) = 0, y\left(\dfrac{5}{2}\right) = \dfrac{1}{5} x\left(\dfrac{1}{2}\right) + \dfrac{2}{7} x\left(\dfrac{5}{4}\right) + \dfrac{3}{8} x\left(\dfrac{7}{4}\right) + \dfrac{4}{11} x\left(\dfrac{9}{4}\right),
\end{cases} \tag{64}
$$

Here, $\psi(t) = t^2 + 1, \alpha = 3/2, \beta = 1/3, p = 4/3, q = 1/2,$

$k = 1/55$, $\nu = 1/58, \gamma = 5/3, \delta = 5/3, \lambda_1 = 1/3, \lambda_2 = 1/6, \lambda_3 = 1/9$, $\mu_1 = 1/5, \mu_2 = 2/7, \mu_3 = 3/8, \mu_4 = 4/11, \theta_1 = 3/4, \theta_2 = 3/2, \theta_3 = 2$, $\zeta_1 = 1/2, \zeta_2 = 5/4, \zeta_3 = 7/4, \zeta_4 = 9/4$, $a = 1/4$, $b = 5/2$, $m = 5$, and $n = 6$.

From the given data, we can calculate $A \approx 3.733460626$, $B \approx 0.8506267338$, $\Omega \approx 2.529197097$, $\Delta \approx 3.733460626$, $\Lambda \approx 11.78732558$, $X_1 \approx 0.09724746240$, $X_2 \approx 0.06772019882$, $Y_1 \approx 0.02206174386$, $Y_2 \approx 0.09253172371$, $F_1 \approx 26.59796061$, $F_2 \approx 15.10628827$, $G_1 \approx 3.858014303$, and $G_2 \approx 21.69467361$.

(i) Let the nonlinear functions f and g be defined on $[1/4, 5/2]$ by

$$f(t, x, y) = \frac{1}{2} e^{-|xy|} + \frac{4}{339 + 4t} \left(\frac{x^2}{1+|x|} \right) + \frac{1}{320t} y \sin^6 x, \tag{65}$$

$$g(t, x, y) = \frac{2}{3} \cos^2 |xy| + \frac{4}{299 + 4t} x e^{-y} + \frac{1}{360t} \left(\frac{y^5}{1 + y^4} \right). \tag{66}$$

It is obvious to check that the above functions satisfy

$$\begin{aligned} |f(t, x, y)| &\le \frac{1}{2} + \frac{1}{85} |x| + \frac{1}{80} |y|, \\ |g(t, x, y)| &\le \frac{2}{3} + \frac{1}{75} |x| + \frac{1}{90} |y|, \end{aligned} \tag{67}$$

which can be set $p_0 = 1/2$, $p_1 = 1/85$, $p_2 = 1/80$, $q_0 = 2/3$, $q_1 = 1/75$, and $q_2 = 1/90$ as in the hypothesis (H_1) of Theorem 13. Then, we can find that

$$\begin{aligned} \mathcal{M}_1 &\approx 0.9963083888 < 1, \\ \mathcal{M}_2 &\approx 0.9198153332 < 1. \end{aligned} \tag{68}$$

Thus, all assumptions of Theorem 13 satisfy. The conclusion of Theorem 13 implies that problem (64) with (65) and (66) has at least one solution on $[1/4, 5/2]$.

(ii) Consider now the functions f and g given by

$$f(t, x, y) = \frac{1}{2} + e^{-3t} + \frac{4}{767 + 4t} \left(\frac{x^2 + 2|x|}{1+|x|} \right) + \frac{1}{392t} \sin |y|, \tag{69}$$

$$g(t, x, y) = \frac{1}{3} + \pi (\log t)^2 + \frac{1}{364t} \tan^{-1} x + \frac{4}{719 + 4t} \left(\frac{y^2 + 2|y|}{1+|y|} \right). \tag{70}$$

Checking the Lipschitz condition for f and g, we obtain

$$\begin{aligned} |f(t, x_1, y_1) - f(t, x_2, y_2)| &\le \frac{1}{96} |x_1 - x_2| + \frac{1}{98} |y_1 - y_2|, \\ |g(t, x_1, y_1) - g(t, x_2, y_2)| &\le \frac{1}{91} |x_1 - x_2| + \frac{1}{90} |y_1 - y_2|. \end{aligned} \tag{71}$$

Then, by setting $\mathcal{P} = 1/96$ and $\mathcal{Q} = 1/90$, the condition (H_2) of Theorem 14 is fulfilled. In addition, we find that

$$[F_1 + F_2]\mathcal{P} + [G_1 + G_2]\mathcal{Q} + [X_1 + X_2] + [Y_1 + Y_2] \approx 0.9978991426 < 1. \tag{72}$$

Therefore, the system (64) with (69) and (70) has a unique solution on $[1/4, 5/2]$, by the benefit of Theorem 14.

4. Conclusion

We investigated the existence and uniqueness of solutions for a coupled system of nonlinear fractional differential equations involving Hilfer fractional derivative with coupled nonlocal multipoint boundary conditions by applying the framework of fixed point theorems. The existence of a unique solution is obtained via Banach's fixed point theorem, while the existence result is proved by using Leray-Schauder alternative. The results obtained in the present paper are new and significantly contribute to the existing literature on the topic.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

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