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Equal and Odd of Generalized Euler Function for Successive Integers

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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Abstract

Euler function $\varphi(n)$ and generalized Euler function $\varphi_e(n)$ are two important functions in number theory. Using the idea of classified discussion and determination of prime types, we study the solutions of odd number of generalized Euler function equations $\varphi_e(n)=\varphi_e(n+1)$ and obtain all the values satisfying the corresponding conditions, where $e=2,3,4,6$.

Keywords: Euler function; generalized Euler function; odd.

1 Introduction

Euler function $\varphi(n)$ is a relatively important in number theory, and it is also studied by the majority of researchers. Euler function $\varphi(n)$ is defined as the number of positive integers not greater than n and relatively prime to n. If $n>1$, let standard factorization of n be $n=p_1^{r_1}p_2^{r_2}...p_k^{r_k}$, where $p_1,p_2,...,p_k$ are different primes, $r_i \ge 1\ (1 \le i \le k)$, then

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$$\varphi(n) = n(1-\frac{1}{p_1})(1-\frac{1}{p_2})\cdots(1-\frac{1}{p_k}) .$$

Generalized Euler function $\varphi_e(n)$ is defined as

$$\varphi_e(n) = \sum_{\substack{i=1 \\ (i,n)=1}}^{\left[\frac{n}{e}\right]} 1 .$$

where $[x]$ is the greatest integer not greater than x, and (i,n) denotes the greatest common divisor of i and n. If $e=1$, the generalized Euler function is just Euler function.

Cai [1,2] studied the parity of $\varphi_e(n)$ when $e=2,3,4,6$, and gives the conditions that both $\varphi_e(n)$ and $\varphi_e(n+1)$ are odd numbers, Liang [3], Cao [4] studied the solutions to the equations involving Euler function, Zhang [5-7] investigated the solutions to two equations involving Euler function $\varphi(n)$ and generalized Euler function $\varphi_2(n)$, Jiang [8] investigated the solutions of generalized Euler function $\varphi_3(n)$.

On page 138 of [9], proposing whether there are infinitely many pairs of consecutive integer pairs n and $n+1$ such that $\varphi(n)=\varphi(n+1)$. Jud McGranie found 1267 values of $\varphi(n)=\varphi(n+1)$ with $n \le 10^{10}$, and the largest of which is $n=9985705185$, $\varphi(n)=\varphi(n+1) = 2^{11}3^5 7\cdot 11$. We find the following theorems on the basics of the fact that the articles [1] and [2] and obtain the solutions of the equation $\varphi_e(n)=\varphi_e(n+1)$ under the condition that both $\varphi_e(n)$ and $\varphi_e(n+1)$ are odd numbers.

Theorem 1.1 Both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd and equal if and only if $n=2$ or 3.

Theorem 1.2 Both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd and equal if and only if $n=3$ or 4 or 5 or 15.

Theorem 1.3 Both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd and equal if and only if $n=4$ or 5 or 6 or 7.

2 Preliminaries

Lemma 2.1[1] Except for $n=2,3,242$, both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd if and only if $n=2p^\beta$, where $\beta \ge 1, p \equiv 3(\mathrm{mod}\,4)$, both $2p^\beta+1$ and p are primes.

Lemma 2.2[1] $\varphi_2(1)=0$, $\varphi_2(2)=1$; when $n \ge 3$, $\varphi_2(n) = \frac{1}{2}\varphi(n)$.

Lemma 2.3[1] Except for $n=3,15,24$, both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd if and only if

1) $n+1 = 2^{2^m}+1 (m \ge 1)$ is prime; or

2) $n=2^q, q \equiv 5(\mathrm{mod}\,6)$, both q and $\frac{2^q+1}{3}$ are primes, where $n=2^q, q \equiv 5(\mathrm{mod}\,6)$, or

3) $n = 3 \cdot 2^{\beta} - 1 (\beta \ge 1)$ is prime.

Lemma 2.4[1] If $n > 3$, $n = 3^{a} \prod_{i=1}^{k} p_i^{a_i}, (p_i, 3) = 1, 1 \le i \le k$, then

$$\varphi_3(n) = \begin{cases} \frac{1}{3}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n)-a-1}}{3}, a = 0 \text{ or } 1, p_i \equiv 2 (\bmod 3), 1 \le i \le k, \\ \frac{1}{3}\varphi(n), \text{ otherwise}, \end{cases}$$

where $\Omega(n)$ is the number of prime factors of n (counting repetitions) and $\omega(n)$ is the number of distinct prime factors of n.

Lemma 2.5[10] For any positive integer m, n, we have

$$\varphi(mn) = \frac{(m,n)\varphi(m)\varphi(n)}{\varphi((m,n))},$$

where (m,n) represents the greatest common divisor of m and n. In particular, when $(m,n) = 1$, we have $\varphi(mn) = \varphi(m)\varphi(n)$.

Lemma 2.6 The value of n such that both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd are listed in Table 1 [2].

Table 1. The value of n such that both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd

n	$n+1$	**conditions**
4	5	
7	8	
57121	57122	
p^2	$2q^2$	$p \equiv 7(\bmod 8), q \equiv 5(\bmod 8)$ are primes
$2q^{\beta} - 1$	$2q^{\beta}$	$2q^{\beta} - 1 \equiv 5(\bmod 8), q \equiv 3(\bmod 8)$ are primes, and β is prime
$2q^{\beta}$	$2q^{\beta} + 1$	$2q^{\beta} + 1 \equiv 7(\bmod 8), q \equiv 3(\bmod 8)$ are primes, and β is prime
p^2	$p^2 + 1$	$p \equiv 5(\bmod 8), \frac{p^2+1}{2} \equiv 5(\bmod 8)$ are primes
$5^{\alpha} - 1$	5^{α}	$\frac{5^{\alpha}-1}{4} \equiv 3(\bmod 4)$ is a prime
$4q^{\beta}$	$4q^{\beta} + 1$	$4q^{\beta} + 1, q \equiv 3(\bmod 4)$ are primes, $\beta \ge 1$

Lemma 2.7 If $n > 4$, $n = 2^{a} \prod_{i=1}^{k} p_i^{a_i}, (p_i, 2) = 1, a \ge 0, 1 \le i \le k$, then [2]

$$\varphi_4(n) = \begin{cases} \frac{1}{4}\varphi(n) + \frac{(-1)^{\Omega(n)} 2^{\omega(n)-a}}{4}, a = 0 \text{ or } 1, p_i \equiv 3 (\bmod 4), 1 \le i \le k, \\ \frac{1}{4}\varphi(n), \text{ otherwise}. \end{cases}$$

3 Proof of the Theorems

3.1 Proof of Theorem 1.1

We have $\varphi_2(2)=\varphi_2(3)=\varphi_2(4)=1$ by definition of the generalized Euler function $\varphi_2(n)$, and $\varphi_2(242)=55,\varphi_2(243)=81$ by Lemma 2.2.

By Lemma 2.1, except for $n=2,3,242$, both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd if and only if $n=2p^\beta$, where $\beta\ge1, p\equiv3(\bmod 4)$, both $2p^\beta+1$ and p are primes. By Lemma 2.2, when $n\ge3,\varphi_2(n)=\frac{1}{2}\varphi(n)$, and $\varphi_2(n+1)=\frac{1}{2}\varphi(n+1)$. Then for the equation $\varphi_e(n)=\varphi_e(n+1)$, we just need to solve the equation

$$\varphi(n)=\varphi(n+1). \tag{1}$$

Put $n=2p^\beta$, $n+1=2p^\beta+1$ in (1), since $n+1=2p^\beta+1$ is prime, then $\varphi(n+1)=n$. We just need to solve the equation

$$\varphi(n)=n,$$

and it has only a solution $n=1$, but the solution is not satisfied with the form $n=2p^\beta$, so there is no solution.

Hence both $\varphi_2(n)$ and $\varphi_2(n+1)$ are odd and equal if and only if $n=2$ or 3.

3.2 Proof of Theorem 1.2

By the definition of $\varphi_3(n)$, We have

$$\varphi_3(3)=1,\varphi_3(4)=1,\varphi_3(15)=3,\varphi_3(16)=3,\varphi_3(24)=3,\varphi_3(25)=7,$$

hence $\varphi_3(3)=\varphi_3(4),\varphi_3(15)=\varphi_3(16)$. Except $n=3,15,24$, we discuss the solutions in 3 cases by Lemma 2.3.

Case 1 When $n=2^{2^m}$, $n+1=2^{2^m}+1(m\ge1)$, and $n+1=2^{2^m}+1(m\ge1)$ is prime. For n , in Lemma2.4, we have $a=0$, $p\equiv2(\bmod 3)$, $\Omega(n)=2^m$, $\omega(n)=1$, then by Lemma 2.4, we have

$$\varphi_3(n)=\frac{1}{3}\varphi(n)+\frac{1}{3}.$$

Since $n+1=2^{2^m}+1$ is prime and $n+1\equiv2(\bmod 3)$, we have

$$\varphi_3(n+1)=\frac{1}{3}\varphi(n+1)-\frac{1}{3}.$$

If $\varphi_3(n)=\varphi_3(n+1)$, then

$$\frac{1}{3}\varphi(n)+\frac{1}{3}=\frac{1}{3}\varphi(n+1)-\frac{1}{3}.$$

Simplify it, we obtain $2^{2^m-1}+1=2^{2^m}-1$, thus we have $m=1$, $n=4$.

Case 2 When $n=2^q, n=2^q+1$, and both $q\equiv 5\ (\mathrm{mod}\ 6)$, $\frac{2^q+1}{3}$ are primes, by Lemma 2.4, we have

$$\varphi_3(n)=\frac{1}{3}\varphi(n)-\frac{1}{3}.$$

Since $\frac{2^q+1}{3}$ is prime , $q\equiv 5\ (\mathrm{mod}\ 6)$ and $\varphi(9)=6$, we have

$$2^q+1\equiv 2^5+1\equiv 33\ (\mathrm{mod}\ 9),$$

thus $\frac{2^q+1}{3}\equiv 11\equiv 2\ (\mathrm{mod}\ 3)$. $n+1=3\times\frac{2^q+1}{3}$, then by Lemma 2.4, we obtain

$$\varphi_3(n+1)=\frac{\varphi(n+1)}{3}+\frac{1}{3}.$$

If $\varphi_3(n)=\varphi_3(n+1)$, then $\varphi(n)=\varphi(n+1)+2$, namely

$$2^q\cdot(1-\frac{1}{2})=2\times(\frac{2^q+1}{3}-1)+2,$$

simplified to $2^q=-4$, we have no solutions in this case.

Case 3 When $n=3\cdot 2^\beta-1$, $n+1=3\cdot 2^\beta$, and $n=3\cdot 2^\beta-1(\beta\ge 1)$ is prime, by Lemma 2.4, we have
$\varphi_3(n)=\frac{1}{3}\varphi(n)-\frac{1}{3}$,

meanwhile,

$$\varphi_3(n+1)=\frac{1}{3}\varphi(n+1)+\frac{(-1)^{1+\beta}2^{\omega(n)-a-1}}{3}=\frac{1}{3}\varphi(n+1)+\frac{(-1)^{1+\beta}}{3}.$$

If $\beta=2k, k>0$

$$\frac{1}{3}\varphi(n)-\frac{1}{3}=\frac{1}{3}\varphi(n+1)-\frac{1}{3},$$

simplified to $\varphi(n)=\varphi(n+1)$. Since $n=3\cdot 2^{\beta}-1(\beta\geq 1)$ is prime, then

$$3\cdot 2^{\beta}-2=3\cdot 2^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3}),$$

We get $\beta=0$, this is contradicted with the condition $\beta\geq 1$. If $\beta=2k+1,k\geq 0$,

$$\frac{1}{3}\varphi(n)-\frac{1}{3}=\frac{1}{3}\varphi(n+1)+\frac{1}{3},$$

simplified to $\varphi(n)=\varphi(n+1)+2$, then

$$3\cdot 2^{\beta}-2=3\cdot 2^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{3})+2,$$

We have $\beta=1$, then $n=3\times 2-1=5$.

Hence, both $\varphi_3(n)$ and $\varphi_3(n+1)$ are odd and equal if and only if $n=3$ or 4 or 5 or 15.

3.3 Proof of Theorem 1.3

By Lemma 2.7, we have $\varphi_4(4)=1,\varphi_4(5)=1$, $\varphi_4(7)=1,\varphi_4(8)=1$ and

$$\varphi_4(57121)=14221,\ \varphi_4(57122)=6591,$$

hence $\varphi_4(4)=\varphi_4(5),\varphi_4(7)=\varphi_4(8).$ Then we discuss the solutions in 6 cases by Lemma 2.6.

Case 1 When $n=p^2,n+1=2q^2$, and both $p\equiv 7(\text{mod}\,8),q\equiv 5(\text{mod}\,8)$ are primes. By Lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)+\frac{1}{2}$. Since $q\equiv 1(\text{mod}\,4)$, then $\varphi_4(n+1)=\frac{1}{4}\varphi(n+1),$ namely

$$\frac{1}{4}\varphi(n)+\frac{1}{2}=\frac{1}{4}\varphi(n+1).$$

Simplified to $\varphi(n)+2=\varphi(n+1),$ namely

$$p^2\cdot(1-\frac{1}{p})+2=2q^2\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q}).$$

Then $q\cdot(q-1)-p\cdot(p-1)=2,$ by $p^2+1\equiv 2q^2$, we have $p=q^2+q+1.$ Then $p^2=(q^2+q+1)^2\geq(q^2+q)^2\geq 36q^2>2q^2,$

which is contradicted with the condition $p^2+1\equiv 2q^2$, no solution.

Case 2 When $n = 2q^{\beta} - 1, n+1 = 2q^{\beta}$, and both $2q^{\beta} - 1 \equiv 5(\bmod 8)$, $q \equiv 3(\bmod 8)$ are primes, where β is an odd. By Lemma 2.7, we have $\varphi_4(n+1) = \frac{1}{4}\varphi(n+1) + \frac{1}{2}$.

Since $2q^{\beta} - 1 \equiv 1(\bmod 4)$, we have $\varphi_4(n) = \frac{1}{4}\varphi(n)$, namely

$$\frac{1}{4}\varphi(n) = \frac{1}{4}\varphi(n+1) + \frac{1}{2}.$$

Simplified to $\varphi(n) = \varphi(n+1) + 2$, namely

$$(2q^{\beta} - 1) - 1 = 2q^{\beta} \cdot (1 - \frac{1}{2}) \cdot (1 - \frac{1}{q}) + 2.$$

Then $(q+1) \cdot q^{\beta-1} = 4$, since both q and $q+1$ are positive integers, and $q \equiv 3(\bmod 8)$, so $q+1 \geq 4$, then $q = 3, \beta = 1$, we have $n = 2 \times 3 - 1 = 5$ such that $\varphi_4(n) = \varphi_4(n+1)$ only in this case.

Case 3 When $n = 2q^{\beta}, n+1 = 2q^{\beta} + 1$, and both $2q^{\beta} + 1 \equiv 7(\bmod 8)$, $q \equiv 3(\bmod 8)$ are primes, where β is an odd. By Lemma 2.7, we have $\varphi_4(n) = \frac{1}{4}\varphi(n) + \frac{1}{2}$ and

$$\varphi_4(n+1) = \frac{1}{4}\varphi(n+1) - \frac{1}{2},$$

Then

$$\frac{1}{4}\varphi(n) + \frac{1}{2} = \frac{1}{4}\varphi(n+1) - \frac{1}{2}.$$

Simplified to $\varphi(n) + 4 = \varphi(n+1)$, namely

$$2q^{\beta} \cdot (1 - \frac{1}{2}) \cdot (1 - \frac{1}{q}) + 4 = 2q^{\beta}.$$

Then $(q+1) \cdot q^{\beta-1} = 4$, since q and $q+1$ both are positive integers, and $q \equiv 3(\bmod 8)$, so $q+1 \geq 4$, then $q = 3, \beta = 1$, we have $n = 2 \times 3 = 6$ such that $\varphi_4(n) = \varphi_4(n+1)$ only in this case.

Case 4 When $n = p^2, n+1 = p^2 + 1$, and both $p \equiv 5(\bmod 8)$, $\frac{p^2+1}{2} \equiv 5(\bmod 8)$ are primes. By Lemma 2.7, we have $\varphi_4(n) = \frac{1}{4}\varphi(n)$ and

$$\varphi_4(n+1) = \frac{1}{4}\varphi(n+1).$$

When $\varphi_4(n)=\varphi_4(n+1)$, we have

$$\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1).$$

Simplified to

$$\mathrm{p}^2\cdot(1-\frac{1}{p})=\frac{p^2+1}{2}-1,$$

then $p=1$. Which contradicts $p\equiv 5(\mathrm{mod}\,8)$.

Case 5 When $n=5^{\alpha}-1, n+1=5^{\alpha}$, and $\frac{5^{\alpha}-1}{4}\equiv 3(\mathrm{mod}\,4)$ is a prime, then $n=4\cdot\frac{5^{\alpha}-1}{4}=2^2\cdot\frac{5^{\alpha}-1}{4}$.

By Lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)$ and

$$\varphi_4(n+1)=\frac{1}{4}\varphi(n+1),$$

namely $\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1)$, simplified to $\varphi(n)=\varphi(n+1)$, i.e., $2\cdot(\frac{5^a-1}{4}-1)=5^a\cdot\frac{4}{5}$,

Then $5^a=-\frac{25}{3}$, which is impossible.

Case 6 When $n=4q^{\beta}, n+1=4q^{\beta}+1$, and both $4q^{\beta}+1, q\equiv 3(\mathrm{mod}\,4)$ are primes, where $\beta\geq 1$.

By Lemma 2.7, we have $\varphi_4(n)=\frac{1}{4}\varphi(n)$ and $\varphi_4(n+1)=\frac{1}{4}\varphi(n+1)$, namely

$$\frac{1}{4}\varphi(n)=\frac{1}{4}\varphi(n+1).$$

Simplified to $\varphi(n)=\varphi(n+1)$, namely

$$4q^{\beta}\cdot(1-\frac{1}{2})\cdot(1-\frac{1}{q})=4q^{\beta}.$$

Then $q=-1$. Which contradicts the condition that $q\equiv 3(\mathrm{mod}\,4)$ is prime.

Hence, both $\varphi_4(n)$ and $\varphi_4(n+1)$ are odd and equal if and only if $n=4$ or 5 or 6 or 7.

4 Conclusion

Euler function $\varphi(n)$ and generalized Euler function $\varphi_e(n)$ are two important functions in number theory. which this article has studied is the odd values of generalized Euler function equation $\varphi_e(n)=\varphi_e(n+1)$,where $e=2,3,4$. Similarly, for $e=6$, we obtain that both $\varphi_6(n)$ and $\varphi_6(n+1)$ are odd and equal if and only if $n=6$ or 7 or 8 or 9 or 10 or 11.

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Competing Interests

Authors have declared that no competing interests exist.

References

[1] Cai Tianxin，Shen Zhongyan，Hu Mengjun. On the Parity of the Generalized Euler Function I [J]. Advances in Mathematics (China). 2013;42 (04): 505-510.

[2] Shen Zhongyan，Cai Tianxin，Hu Mengjun. On the Parity of the Generalized Euler Function (II)[J]. Advances in Mathematics (China). 2016;45(4):509-519.

[3] Liang Xiaoyan. A Research on the Solutions to Equation of Euler Function and Pseudo Smarandache Function [D].Yan'an University;2021.

[4] Cao Panpan,Zhao Xiqing. Positive Integer Solutions of Generalized Euler Function Equation $\varphi_2(n)=S(n^{28})$ [J].Journal of Yanan University (Natural Science Edition). 2020; 39(04):72-76.

[5] Zhang Sibao. Solutions of two equations related to generalized Euler function [J].Journal of Anhui University (Natural Science Edition). 2019;43(04): 32-35.

[6] Zhang Sibao,Akemu Youlidaxi. Solutions of several equations mixing Euler functions with generalized Euler functions[J].Journal of Northeast Normal University (Natural Science Edition). 2020;52(01):17-20.

[7] Zhang Sibao,Guan Chunmei,Yang Yanni. An Equation on Euler Function and Generalized Euler Function[J]. Mathematics in Practice and Theory. 2018; 48(09): 265-268.

[8] Jiang Lianxia,Zhang Sibao. Solution of an equation on generalized Euler's function $\varphi_3(n)$ [J].Journal of Capital Normal University (Natural Science Edition). 2020; 41(06):1-5.

[9] Richard K. Guy. Unsolved Problems in Number Theory (Third Edition) [M].Spring Science Business Media,Inc;2007.

[10] Zhang Wenpeng. Elementary Number Theory [M]. Xi'an:Shanxi Normal University Press; 2007.

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