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Stability Analysis of Perturbed Linear Non-integer Differential Systems

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Author's contribution

The sole author designed, analyzed, interpreted and prepared the manuscript.

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Abstract

In this work, the effect of perturbation on linear fractional differential system is studied. The analysis is done using Riemann-Liouville derivative and the conclusion extended to using Caputo derivative since the result is similar. Conditions for determining the stability and asymptotic stability of perturbed linear fractional differential system are given.

Keywords: Asymptotic stability; riemann-liouville derivative; caputo derivative; perturbed fractional differential systems.

1 Introduction

Fractional calculus has attracted increasing interest in the last three decades due to the fact that many mathematical problems in sciences and engineering can be modeled as fractional differential equations. Fractional differential equations have found many applications in physics, control engineering and signal processing. In interdisciplinary fields, many systems can be elegantly described with the help of the fractional

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derivatives. Stability analysis is basic in the study of fractional differential systems due to its importance. This analysis helps to overcome unnecessary destruction or destabilization in dealing with an unstabilized systems.

Many studies have been made on the stability of linear fractional differential systems[1-11]. The need to extend the study of linear fractional differential systems to perturbed fractional differential systems has attracted the interest of researchers in recent times. [12-15] studied perturbed fractional differential systems and gave stability conditions for the stability and asymptotic stability of fractional differential systems. This work seeks to add to knowledge in the study of perturbed differential systems. Some results have been established which are used in determining the stability or otherwise of the systems. Section 2 of this work gives the preliminaries and definitions while the stability analysis is given in section 3. Conclusion is given in section 4.

2 Preliminaries and Definitions

In this section, the basic definitions and concepts are given. These results and definitions will be used in the analysis that follows.

Definition 2.1 (Gamma Function): Gamma function is the generalization of the factorial function to non-integral values, introduced by the Swiss mathematician Leonhard Euler in the 18th century. The gamma function represented by Γ (the capital letter gamma from the Greek alphabet) is one commonly used extension of the fractional function to complex numbers. The gamma function is defined for all complex numbers except the non-positive integer. For any positive integer n, $\Gamma(n) = (n-1)!$. But this formula is meaningless if n is not an integer. To extend the factorial to any real number $x > 0$ (whether or not x is a whole number), the gamma function is defined as

$$\Gamma(x) = \int_0^\infty t^{x-1} e^{-t}\, dt \qquad (x > 0)$$

Definition 2.2: The Riemann-Liouville derivative and the Caputo derivative will be used in the analysis.

The Riemann-Liouville derivative is defined as

$${}_{RL}D_{a,t}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \left(\frac{d}{dt}\right)^n \int_0^\infty (t-\tau)^{n-\alpha-1} x(\tau) d\tau, \qquad (n-1 \le \alpha < n)$$

And the Caputo derivative is defined as

$${}_{C}D_{a,t}^{\alpha} x(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^\infty (t-\tau)^{n-\alpha-1} x^{(n)}(\tau) d\tau, \qquad (n-1 \le \alpha < n)$$

where $\Gamma(.)\ is\ the\ Euler's\ integral (gamma\ function)$.

The Laplace transform of the Riemann-Liouville fractional derivative ${}_{RL}D_{a,t}^{\alpha} x(t)$ is given as

$$\int_0^\infty e^{-st}\, {}_{RL}D_{a,t}^{\alpha} x(t) dt = S^\alpha X(s) - \sum_{k=0}^{n-1} S^k \left[D^{\alpha-k-1} x(t)\right]_{t=a} \qquad (n-1 \le \alpha < n)$$

Similarly, the Laplace transform of the Caputo fractional differential derivative ${}_{C}D_{a,t}^{\alpha} x(t)$ is given as

$$\int_0^\infty e^{-st}\, {}_{C}D_{a,t}^{\alpha} x(t) dt = S^\alpha X(s) - \sum_{k=0}^{n-1} S^{\alpha-k-1} x^{(k)}(a) , \qquad (n-1 \le \alpha < n)$$

Definition 2.3: The Mittag-Leffler function is defined by

$$E_\alpha(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha+1)} , \qquad where\ Re(\alpha) > 0,\ z \epsilon C$$

The two parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(k\alpha + \beta)} \quad , \qquad (\alpha > 0,\ \ \beta > 0)$$

The Laplace transform of the Mittag-Leffler function is given as

$$\int_0^{\infty} e^{-st} t^{ak-\beta-1} E_{\alpha,\beta}^{(k)}(\pm a t^{\alpha}) dt = \frac{k!\, S^{\alpha-\beta}}{(S^{\alpha} \mp a)^{k+1}} \ \ ; (R(s) > |a|^{\frac{1}{n}})$$

Proposition 2.1: If $A \in C^{n\times n}$ *and* $0 < \alpha < 2$ *,* β *is an arbitrary complex number and* σ *satisfies* $\frac{\alpha\pi}{2} < \sigma < \min(\pi, \alpha\pi)$ *, then for an arbitrary integer* $p \ \geq 1$*, the following expansions hold*:

$$E_{\alpha,\beta}(z) = \frac{1}{\alpha} z^{1-\beta/\alpha} \exp\left(z^{1/\alpha}\right) - \sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|a|^{-p-1}) \ \ ,$$

with $|z| \to \infty, |\arg(z)| \leq \sigma$

and

$$E_{\alpha,\beta}(z) = -\sum_{k=1}^{p} \frac{z^{-k}}{\Gamma(\beta - k\alpha)} + O(|a|^{-p-1}) \ \ ,$$

with $|z| \to \infty$ *and* $\sigma < |\arg(z)| \leq \pi$

Proposition 2.2: Suppose $\alpha > 0$, $a(t)$ is a nonnegative locally integrable function on $0 \leq t < T$(*some* $T \leq \infty$ *and* gt is a nonnegative and nondecreasing continuous function defined on *0≤t<T , gt≤Mconstant, and suppose* $u(t)$*is nonnegative and locally integrable on* $0 \leq t < T$ *with*

$$u(t) \leq a(t) + g(t) \int_0^t (t-s)^{\alpha-1} u(s)\, ds$$

on this interval, then

$$u(t) \leq a(t) + \int_0^t [\sum_{k=0}^{\infty} \frac{(g(t)\Gamma(\alpha))^k}{\Gamma(n\alpha)} \ (t-s)^{n\alpha-1} a(s)] ds$$

Also, if *a(t)* is a nondecreasing function on [0,T), then

$$u(t) \leq a(t) E_{\alpha}(g(t)\Gamma(\alpha) t^{\alpha}).$$

3 Stability Analysis

Consider the perturbed system given by

$${}_{RL}D_{a,t}^{\alpha} x(t) = Ax(t) + f\big(t, x(t)\big), \qquad t > a \tag{3.1}$$

with the initial condition

$${}_{RL}D_{a,t}^{\alpha-k} x(t) = x_{k-1} \tag{3.2}$$

Or

$$ {}_{C}D_{a,t}^{\alpha}x(t) = Ax(t) + f\big(t, x(t)\big), \qquad t > a \tag{3.3} $$

with the initial condition

$$ {}_{C}D_{a,t}^{\alpha-k}x(t) = x_{k-1} \tag{3.4} $$

where $x \in R^n$, $matrix\ P \in R^{n\times n}$ $\ and\ $ $1 < \alpha < 2.$

$f(t,x)\colon [t,\infty) \times R^n \to R^n$ is a continuous function and $f(t,x)$ satisfies the Lipschitz condition with respect to x.

It is pertinent to mention that the solutions analysis gives the same result. In this work, (3.1) with (3.2) is used for the analysis and the conclusion is extended to (3.3) with (3.4).

The solution of(3.1) with (3.2) is given by

$$ \begin{aligned} x(t) = (t-a)^{\alpha-1}E_{\alpha,\alpha}(P(t-a)^{\alpha})x_0 + (t-a)^{\alpha-2}E_{\alpha,\alpha-1}(P(t-a)^{\alpha})x_1 \\ + \int_a^t (t-\tau)^{\alpha-1}\, E_{\alpha,\alpha}(P(t-\tau)^{\alpha}) f(\tau, x(\tau)) x(\tau) d\tau \end{aligned} $$

Applying the norm, we have

$$ \begin{aligned} \|x(t)\| \le \left\|(t-a)^{\alpha-1}E_{\alpha,\alpha}(P(t-a)^{\alpha})\right\| \|x_0\| + \left\|(t-a)^{\alpha-2}E_{\alpha,\alpha-1}(P(t-a)^{\alpha})\right\| \|x_1\| \\ + \int_a^t (t-\tau)^{\alpha-1} \left\|E_{\alpha,\alpha}(P(t-\tau)^{\alpha})\right\| \times \|f(\tau, x(\tau))\| \|x(\tau)\| d\tau \end{aligned} $$

We estimate as follows:

$$ \left\|(t-a)^{\alpha-1}E_{\alpha,\alpha}(P(t-a)^{\alpha})\right\| \le M_0 \ , \ \left\|(t-a)^{\alpha-2}E_{\alpha,\alpha-1}(P(t-a)^{\alpha})\right\| \le M_1 $$

$$ \left\|E_{\alpha,\alpha}(P(t-\tau)^{\alpha})\right\| \le L \ , \qquad \|f(\tau, x(\tau))\| \le M $$

Using the above estimates, we have

$$ \|x(t)\| \le M_0\|x_0\| + M_1\|x_1\| + LM \int_a^t (t-\tau)^{\alpha-1} \|x(\tau)\| d\tau $$

From Proposition 2.1 and Proposition 2.2 , we have the following

$$ \|x(t)\| \le (M_0\|x_0\| + M_1\|x_1\|) E_{\alpha}(LM\Gamma(\alpha)(t-a)^{\alpha}) $$

$$ \|x(t)\| = (M_0\|x_0\| + M_1\|x_1\|) \times \left[-\sum_{k=1}^{p} \frac{(LM\Gamma(\alpha)(t-a)^{\alpha})^{-k}}{\Gamma(1-k\alpha)} + O(LM\Gamma(\alpha)(t)^{\alpha})^{-1-q} \right] $$

when $t \to \infty$, $\|x(t)\| \to 0$. Therefore, if the eigenvalues of P satisfy $|\arg(\lambda(P))| > \frac{\alpha\pi}{2}$, then the solution of (3.1) with (3.2) is asymptotically stable.

To examine the situation where the solution is stable but not asymptotically stable, the following theorem is stated.

Theorem: If the matrix P such that $|spec(P)| \neq 0$, $|\arg(spec(P))| \geq \frac{\alpha\pi}{2}$, the critical eigenvalues which satisfy $|\arg(spec(P))| = \frac{\alpha\pi}{2}$ have the same algebraic and geometric multiplicities. Suppose that there exists a positive function $\gamma(t)$ such that $\int_0^\infty \gamma(t)dt$ is bounded and $f(t, x)$ satisfies Lipschitz condition.

4 Conclusion

The need to ensure or maintain stability of systems has been of immense interest to scientists and engineers. Perturbation is known to cause changes in systems. In this work, the analysis of perturbed system is done using Riemann-Liouville derivative and Caputo derivative. The result of the analysis using both derivatives is the same. The conditions for the determination of the stability and asymptotic stability of the perturbed systems have been provided using classical results and concepts.

Competing Interests

Author has declared that no competing interests exist.

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