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Some Properties of Generalized Fibonacci Numbers: Identities, Recurrence Properties and Closed Forms of the Sum Formulas $\sum_{k=0}^{n} x^k W_{mk+j}$

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Author's contribution

The sole author designed, analysed, interpreted and prepared the manuscript.

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ABSTRACT

In this paper, closed forms of the summation formulas $\sum_{k=0}^{n} x^k W_{mk+j}$ for generalized Fibonacci numbers are presented. As special cases, we give summation formulas of Fibonacci, Lucas, Pell, Pell-Lucas, Jacobsthal, Jacobsthal-Lucas numbers. We present the proofs to indicate how these formulas, in general, were discovered. Of course, all the listed formulas may be proved by induction, but that method of proof gives no clue about their discovery. Moreover, we give some identities and recurrence properties of generalized Fibonacci sequence.

Keywords: Fibonacci numbers; Lucas numbers; Pell numbers; Jacobsthal numbers; sum formulas; recurrence properties.

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1 INTRODUCTION

The generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) $\{W_n(W_0,W_1;r,s)\}_{n\geq 0}$ (or shortly $\{W_n\}_{n\geq 0}$) is defined (by Horadam [1]) as follows:

$$W_n = rW_{n-1} + sW_{n-2}, \quad W_0 = a, W_1 = b, \;\; n \geq 2 \quad (1.1)$$

where W_0, W_1 are arbitrary complex (or real) numbers and r,s are real numbers, see also Horadam [2],[3],[4] and Soykan [5].

The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = -\frac{r}{s}W_{-(n-1)} + \frac{1}{s}W_{-(n-2)}$$

for $n = 1,2,3,...$ when $s \neq 0$. Therefore, recurrence (1.1) holds for all integer n.

For some specific values of a,b,r and s, it is worth presenting these special Horadam numbers in a table as a specific name. In literature, for example, the following names and notations (see Table 1) are used for the special cases of r,s and initial values.

Here, OEIS stands for On-line Encyclopedia of Integer Sequences.

Jacobsthal sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [7],[8],[9],[10],[11],[12],[13],[14],[15],[16],[17],[18],[19],[20],[21].

Pell sequence has been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see for example, [22],[23],[24],[25],[26],[27],[28],[29]. For higher order Pell sequences, see [30],[31],[32],[33],[34],[35].

Table 1. A few special case of generalized Fibonacci sequences

Name of sequence	$W_n(a,b;r,s)$	Binet Formula	OEIS [6]
Fibonacci	$W_n(0,1;1,1) = F_n$	$\frac{\left(\frac{1+\sqrt{5}}{2}\right)^n - \left(\frac{1-\sqrt{5}}{2}\right)^n}{\sqrt{5}}$	A000045
Lucas	$W_n(2,1;1,1) = L_n$	$\left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n$	A000032
Pell	$W_n(0,1;2,1) = P_n$	$\frac{(1+\sqrt{2})^n - (1-\sqrt{2})^n}{2\sqrt{2}}$	A000129
Pell-Lucas	$W_n(2,2;2,1) = Q_n$	$(1+\sqrt{2})^n + (1-\sqrt{2})^n$	A002203
Jacobsthal	$W_n(0,1;1,2) = J_n$	$\frac{2^n-(-1)^n}{3}$	A001045
Jacobsthal-Lucas	$W_n(2,1;1,2) = j_n$	$2^n + (-1)^n$	A014551

Now we define two special cases of the sequence $\{W_n\}$. (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are defined, respectively, by the second-order recurrence relations

$$G_{n+2} = rG_{n+1} + sG_n, \quad G_0 = 0, G_1 = 1, \quad (1.2)$$

$$H_{n+2} = rH_{n+1} + sH_n, \quad H_0 = 2, H_1 = r, \quad (1.3)$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$ and $\{E_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$G_{-n} = -\frac{r}{s}G_{-(n-1)} + \frac{1}{s}G_{-(n-2)},$$

$$H_{-n} = -\frac{r}{s}H_{-(n-1)} + \frac{1}{s}H_{-(n-2)},$$

for $n = 1,2,3,...$ respectively. Therefore, recurrences (1.2)-(1.3) hold for all integer n.

Some special cases of (r,s) sequence $\{G_n(0,1;r,s)\}_{n\geq 0}$ and Lucas (r,s) sequence $\{H_n(2,r;r,s)\}_{n\geq 0}$ are as follows:

1. $G_n(0,1;1,1) = F_n$, Fibonacci sequence,
2. $H_n(2,1;1,1) = L_n$, Lucas sequence,
3. $G_n(0,1;2,1) = P_n$, Pell sequence,
4. $H_n(2,2;2,1) = Q_n$, Pell-Lucas sequence,
5. $G_n(0,1;1,2) = J_n$, Jacobsthal sequence,
6. $H_n(2,1;1,2) = j_n$, Jacobsthal-Lucas sequence.

We give the ordinary generating function $\sum\limits_{n=0}^{\infty} W_n x^n$ of the sequence $\{W_n\}$.

Lemma 1.1. *Suppose that $f_{W_n}(x) = \sum\limits_{n=0}^{\infty} W_n x^n$ is the ordinary generating function of the generalized Fibonacci sequence $\{W_n\}_{n\geq 0}$. Then, $\sum\limits_{n=0}^{\infty} W_n x^n$ is given by*

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - rW_0)x}{1 - rx - sx^2}. \tag{1.4}$$

Binet's formula of generalized Fibonacci sequence can be calculated using its characteristic equation (the quadratic equation) which is given as

$$x^2 - rx - s = 0. \tag{1.5}$$

The roots of characteristic equation are

$$\alpha = \frac{r+\sqrt{\Delta}}{2}, \;\; \beta = \frac{r-\sqrt{\Delta}}{2}. \tag{1.6}$$

where

$$\Delta = r^2 + 4s$$

and the followings hold

$$\begin{array}{rcl} \alpha + \beta &=& r, \\ \alpha\beta &=& -s, \\ (\alpha - \beta)^2 &=& (\alpha+\beta)^2 - 4\alpha\beta = r^2 + 4s. \end{array}$$

1.1 Binet's Formula for the Distinct Roots Case

In this subsection, we assume that the roots α and β of characteristic equation (1.5) are distinct. Using these roots and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.2 (Distinct Roots Case)**.** *Binet's formula of generalized Fibonacci numbers is*

$$W_n = \frac{b_1\alpha^n}{\alpha - \beta} + \frac{b_2\beta^n}{\beta - \alpha} = \frac{b_1\alpha^n - b_2\beta^n}{\alpha - \beta} \tag{1.7}$$

where

$$b_1 = W_1 - \beta W_0, \; b_2 = W_1 - \alpha W_0.$$

(1.7) can be written in the following form:

$$W_n = A_1\alpha^n + A_2\beta^n \tag{1.8}$$

where

$$A_1 = \frac{W_1 - \beta W_0}{\alpha - \beta},\ A_2 = \frac{W_1 - \alpha W_0}{\beta - \alpha}.$$

Note that

$$\begin{aligned} A_1A_2 &= \frac{(W_1^2 - sW_0^2 - rW_1W_0)}{-(r^2+4s)}, \\ A_1 + A_2 &= W_0. \end{aligned}$$

We next find Binet's formula of generalized Fibonacci numbers $\{W_n\}$ by the use of generating function for W_n.

Theorem 1.3. *(Binet's formula of generalized Fibonacci numbers)*

$$W_n = \frac{d_1\alpha^n}{(\alpha - \beta)} + \frac{d_2\beta^n}{(\beta - \alpha)} \tag{1.9}$$

where

$$\begin{aligned} d_1 &= W_0\alpha + (W_1 - rW_0), \\ d_2 &= W_0\beta + (W_1 - rW_0)\beta. \end{aligned}$$

Proof. For a proof see [5], Theorem 1.2]. □

Note that from (1.7) and (1.9) we have

$$W_1 - \beta W_0 = W_0\alpha + (W_1 - rW_0), \tag{1.10}$$

$$W_1 - \alpha W_0 = W_0\beta + (W_1 - rW_0)\beta. \tag{1.11}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.7) or (1.9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= \frac{\alpha^n}{(\alpha - \beta)} + \frac{\beta^n}{(\beta - \alpha)}, \\ H_n &= \alpha^n + \beta^n, \end{aligned}$$

respectively.

1.2 Binet's Formula for the Single Root Case

In this subsection, we assume that the roots α and β of characteristic equation (1.5) are equal, i.e., $\alpha = \beta$. So (1.5) can be written as

$$x^2 - rx - s = (x - \alpha)^2 = x^2 - 2\alpha x + \alpha^2 = 0.$$

Note that in this case,

$$\begin{aligned} \alpha &= \frac{r}{2}, \\ r &= 2\alpha, \\ s &= -\alpha^2 = -\frac{r^2}{4}, \\ r^2 + 4s &= 0. \end{aligned}$$

Using the root α and the recurrence relation, Binet's formula can be given as follows:

Theorem 1.4 (Single Root Case)**.** *Binet's formula of generalized Fibonacci numbers is*

$$W_n = (D_1 + D_2 n)\alpha^n \tag{1.12}$$

where

$$\begin{aligned} D_1 &= W_0, \\ D_2 &= \frac{1}{\alpha}\left(W_1 - \alpha W_0\right). \end{aligned}$$

Proof. W_n is in the following form:

$$W_n = (D_1 + D_2 \times n)\alpha^n$$

where D_1 and D_2 are the numbers whose values are determined by the values W_0 and any other known value of the sequence. By using the values W_0 and W_1, we obtain

$$\begin{aligned} W_0 &= (D_1 + D_2 \times 0)\alpha^0 \\ W_1 &= (D_1 + D_2 \times 1)\alpha^1. \end{aligned}$$

Solving these two simultaneous equations for W_0 and W_1, we get

$$D_1 = W_0, D_2 = \frac{1}{\alpha}\left(W_1 - \alpha W_0\right). \ \square$$

Note that (1.12) can be written as

$$W_n = \left(nW_1 - \frac{r}{2}\left(n-1\right)W_0\right)\left(\frac{r}{2}\right)^{n-1}$$

Note also that

$$\begin{aligned} D_1 D_2 &= \frac{W_0(2W_1 - rW_0)}{r}, \\ D_1 + D_2 &= 2\frac{W_1}{r}. \end{aligned}$$

For all integers n, (r,s) and Lucas (r,s) numbers (using initial conditions in (1.7) or (1.9)) can be expressed using Binet's formulas as

$$\begin{aligned} G_n &= n\alpha^{n-1}, \\ H_n &= 2\alpha^n, \end{aligned}$$

respectively.

2 SOME IDENTITIES

In this section, we obtain some identities of (r,s) and Lucas (r,s) numbers. Firstly, we can give a few basic relations between $\{G_n\}$ and $\{W_n\}$.

Lemma 2.1. *The following equalities are true:*

$$\begin{aligned} s^3 W_n &= ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+4} + (-r(2s+r^2)W_1 + (3r^2 s + r^4 + s^2)W_0)G_{n+3}, \\ s^2 W_n &= (-W_1 r + (r^2+s)W_0)G_{n+3} + ((s+r^2)W_1 - r(2s+r^2)W_0)G_{n+2}, \\ sW_n &= (W_1 - rW_0)G_{n+2} + (-rW_1 + (s+r^2)W_0)G_{n+1}, \\ W_n &= W_0 G_{n+1} + (W_1 - rW_0)G_n, \\ W_n &= W_1 G_n + sW_0 G_{n-1}, \end{aligned}$$

and

$$
\begin{array}{rcl}
s^3(-W_1^2+sW_0^2+rW_1W_0)G_n &=& -((s+r^2)W_1+srW_0)W_{n+4}+(r(2s+r^2)W_1+s(s+r^2)W_0)W_{n+3},\\
s^2(-W_1^2+sW_0^2+rW_1W_0)G_n &=& (rW_1+sW_0)W_{n+3}-((s+r^2)W_1+srW_0)W_{n+2},\\
s(-W_1^2+sW_0^2+rW_1W_0)G_n &=& -W_1W_{n+2}+(rW_1+sW_0)W_{n+1},\\
(-W_1^2+sW_0^2+rW_1W_0)G_n &=& W_0W_{n+1}-W_1W_n,\\
(-W_1^2+sW_0^2+rW_1W_0)G_n &=& -(W_1-rW_0)W_n+sW_0W_{n-1}.
\end{array}
$$

Proof. We prove (a). Writing

$$W_n = a\times G_{n+4}+b\times G_{n+3}$$

and solving the system of equations

$$
\begin{array}{rcl}
W_0 &=& a\times G_4+b\times G_3\\
W_1 &=& a\times G_5+b\times G_4
\end{array}
$$

we find that $a=\frac{(s+r^2)W_1-r(2s+r^2)W_0}{s^3}$, $b=\frac{-r(2s+r^2)W_1+(3r^2s+r^4+s^2)W_0}{s^3}$. The other equalities can be proved similarly. □

Note that all the identities in the above Lemma can be proved by induction as well.

Secondly, we can give a few basic relations between $\{H_n\}$ and $\{W_n\}$.

Lemma 2.2. *The following equalities are true:*

$$
\begin{array}{rcl}
s^3(4s+r^2)W_n &=& (-r(3s+r^2)W_1+(4r^2s+r^4+2s^2)W_0)H_{n+4}+((r^4+2s^2+4r^2s)W_1\\
&& -r(5r^2s+r^4+5s^2)W_0)H_{n+3},\\
s^2(4s+r^2)W_n &=& ((2s+r^2)W_1-r(3s+r^2)W_0)H_{n+3}+(-r(3s+r^2)W_1+(r^4+2s^2+4r^2s)W_0)H_{n+2},\\
s(4s+r^2)W_n &=& (-rW_1+(2s+r^2)W_0)H_{n+2}+((2s+r^2)W_1-r(3s+r^2)W_0)H_{n+1},\\
(4s+r^2)W_n &=& (2W_1-rW_0)H_{n+1}+(-rW_1+(2s+r^2)W_0)H_n,\\
(4s+r^2)W_n &=& (rW_1+2sW_0)H_n+s(2W_1-rW_0)H_{n-1},
\end{array}
$$

and

$$
\begin{array}{rcl}
s^3(-W_1^2+sW_0^2+rW_0W_1)H_n &=& (r(3s+r^2)W_1+s(2s+r^2)W_0)W_{n+4}-((r^4+2s^2+4r^2s)W_1\\
&& +rs(3s+r^2)W_0)W_{n+3},\\
s^2(-W_1^2+sW_0^2+rW_0W_1)H_n &=& -((2s+r^2)W_1+rsW_0)W_{n+3}+(r(3s+r^2)W_1+s(2s+r^2)W_0)W_{n+2},\\
s(-W_1^2+sW_0^2+rW_0W_1)H_n &=& (rW_1+2sW_0)W_{n+2}-((2s+r^2)W_1+rsW_0)W_{n+1},\\
(-W_1^2+sW_0^2+rW_0W_1)H_n &=& (-2W_1+rW_0)W_{n+1}+(rW_1+2sW_0)W_n,\\
(-W_1^2+sW_0^2+rW_0W_1)H_n &=& (-rW_1+(2s+r^2)W_0)W_n+s(-2W_1+rW_0)W_{n-1}.
\end{array}
$$

3 ON THE RECURRENCE PROPERTIES OF GENERALIZED FIBONACCI SEQUENCE

Horadam [36] give the following identity for the second order recurrence relation (1.1).

Theorem 3.1. *For* $n\in\mathbb{Z}$, *we have*

$$W_{n+2k}=H_kW_{n+k}+(-1)^{k+1}s^kW_n.$$

Now, we can propose a problem as follows: Whether and how can the generalized Fibonacci sequence W_n at negative indices be expressed by the sequence itself at positive indices?

We present the following result which completely solves the above problem for the generalized Fibonacci sequence W_n.

Theorem 3.2. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r, s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have*

$$\begin{aligned} W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0) \\ &= (-1)^{n+1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

Proof. If the roots of characteristic equation (1.5) are distinct then by using the Binet's formulas of W_n and H_n we get

$$\begin{aligned} (-1)^{n+1}s^nW_{-n} &= -(-s)^nW_{-n} \\ &= -\alpha^n\beta^n(A_1\alpha^{-n} + A_2\beta^{-n}) \\ &= -(\beta^nA_1 + \alpha^nA_2) \\ &= (A_1\alpha^n + A_2\beta^n) - (A_1 + A_2)(\alpha^n + \beta^n) \\ &= W_n - W_0H_n \\ &\Rightarrow \\ W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

and if the roots of characteristic equation (1.5) are equal then by using the Binet's formulas of W_n and H_n we obtain

$$\begin{aligned} (-1)^{n+1}s^nW_{-n} &= -(-s)^nW_{-n} \\ &= -\alpha^{2n}(D_1 + D_2 \times (-n))\alpha^{-n} \\ &= -\alpha^{2n}(W_0 + \frac{1}{\alpha}(W_1 - \alpha W_0) \times (-n))\alpha^{-n} \\ &= (nW_1 - \alpha(n-1)W_0)\alpha^{n-1} - W_0 \times 2\alpha^n \\ &= (D_1 + D_2 \times n)\alpha^n - W_0 \times 2\alpha^n \\ &= W_n - W_0H_n \\ &\Rightarrow \\ W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

This proves the theorem.

We can obtain the same result by using Theorem 3.1 as follows:

$$\begin{aligned} W_{n+2k} &= H_kW_{n+k} + (-1)^{k+1}s^kW_n \\ &\Rightarrow \\ &\quad \text{by taking } -n \text{ and } n \text{ for } n \text{ and } k \text{ respectively} \\ W_{-n+2n} &= H_nW_{-n+n} + (-1)^{n+1}s^nW_{-n} \\ &\Rightarrow \\ W_n &= H_nW_0 + (-1)^{n+1}s^nW_{-n} \\ &\Rightarrow \\ (-1)^{n+1}s^nW_{-n} &= W_n - H_nW_0 \\ &\Rightarrow \\ W_{-n} &= (-1)^{-n-1}s^{-n}(W_n - H_nW_0). \end{aligned}$$

□

Note that from the definition of H_n, we obtain

$$H_{-n} = (-s)^{-n} H_n$$

i.e., $H_{-n} = (-s)^{-n} H_n$ and so $H_n = (-s)^n H_{-n}$. Note also that

$$(-s)^n = \frac{1}{2}(H_n^2 - H_{2n}).$$

By using Lemma 2.2 and Theorem 3.2 we obtain the following theorem.

Theorem 3.3. *For $n \in \mathbb{Z}$, for the generalized Fibonacci sequence (or generalized (r,s)-sequence or Horadam sequence or 2-step Fibonacci sequence) we have*

$$W_{-n} = \frac{(-1)^{n+1}s^{-n}}{-W_1^2 + sW_0^2 + rW_0W_1}((2W_1 - rW_0)W_0W_{n+1} - (W_1^2 + sW_0^2)W_n).$$

Taking $r = 1, s = 1$ in Theorem 3.2 and Theorem 3.3, we obtain the following Proposition.

Proposition 3.1. *For $n \in \mathbb{Z}$, generalized Fibonacci numbers (the case $r = 1, s = 1$) have the following identity:*

$$\begin{aligned} W_{-n} &= (-1)^{n+1}(W_n - L_nW_0) \\ &= \frac{(-1)^{n+1}}{-W_1^2 + W_0^2 + W_0W_1}((2W_1 - W_0)W_0W_{n+1} - (W_1^2 + W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized Fibonacci sequence at the positive index and the negative index: for Fibonacci and Lucas numbers, take

$W_n = F_n$ with $F_0 = 0, F_1 = 1$ and take $W_n = L_n$ with $L_0 = 2, L_1 = 1$, respectively. Note that in this case $H_n = L_n$.

Corollary 3.4. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Fibonacci sequence:*
$F_{-n} = (-1)^{n+1}F_n$.

(b) *Fibonacci-Lucas sequence:*
$L_{-n} = (-1)^n L_n$.

Taking $r = 2, s = 1$ in Theorem 3.2 and Theorem 3.3, we obtain the following Proposition.

Proposition 3.2. *For $n \in \mathbb{Z}$, generalized Pell numbers (the case $r = 2, s = 1$) have the following identity:*

$$\begin{aligned} W_{-n} &= (-1)^{n+1}(W_n - Q_nW_0) \\ &= \frac{(-1)^{n+1}}{-W_1^2 + W_0^2 + 2W_0W_1}((2W_1 - 2W_0)W_0W_{n+1} - (W_1^2 + W_0^2)W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized Pell sequence at the positive index and the negative index: for Pell and Pell-Lucas numbers, take
$W_n = P_n$ with $P_0 = 0, P_1 = 1$ and take $W_n = Q_n$ with $Q_0 = 2, Q_1 = 2$, respectively. Note that in this case $H_n = Q_n$.

Corollary 3.5. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Pell sequence:*
$P_{-n} = (-1)^{n+1} P_n.$

(b) *Pell-Lucas sequence:*
$Q_{-n} = (-1)^n Q_n.$

Taking $r = 1, s = 2$ in Theorem 3.2 and Theorem 3.3, we obtain the following Proposition.

Proposition 3.3. *For $n \in \mathbb{Z}$, generalized Jacobsthal numbers (the case $r = 1, s = 2$) have the following identity:*

$$\begin{aligned} W_{-n} &= (-1)^{n+1} 2^{-n} (W_n - j_n W_0) \\ &= \frac{(-1)^{n+1} 2^{-n}}{-W_1^2 + 2W_0^2 + W_0 W_1} ((2W_1 - W_0) W_0 W_{n+1} - (W_1^2 + 2W_0^2) W_n). \end{aligned}$$

From the above Proposition, we have the following corollary which gives the connection between the special cases of generalized Jacobsthal sequence at the positive index and the negative index: for Jacobsthal and Jacobsthal-Lucas numbers, take

$W_n = J_n$ with $J_0 = 0, J_1 = 1$ and take $W_n = j_n$ with $j_0 = 2, j_1 = 1$, respectively. Note that in this case $H_n = j_n$.

Corollary 3.6. *For $n \in \mathbb{Z}$, we have the following recurrence relations:*

(a) *Jacobsthal sequence:*
$J_{-n} = (-1)^{n+1} 2^{-n} J_n.$

(b) *Jacobsthal-Lucas sequence:*
$j_{-n} = (-1)^n 2^{-n} j_n.$

4 THE SUM FORMULA $\sum_{k=0}^{n} x^k W_{mk+j}$

In this section, we present sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers).

The following theorem presents sum formulas of generalized (r, s) numbers (generalized Fibonacci numbers).

Theorem 4.1. *Let x be a real (or complex) number. For all integers m and j, for generalized (r, s) numbers (generalized Fibonacci numbers), we have the following sum formulas:*

(a) *If $(-s)^m x^2 - xH_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m) x^{n+1} W_{mn+j} + (-s)^m x^{n+1} W_{mn+j-m} + W_j - (-s)^m x W_{j-m}}{(-s)^m x^2 - xH_m + 1}. \tag{4.1}$$

(b) *If $(-s)^m x^2 - xH_m + 1 = u(x-a)(x-b) = 0$ for some $u, a, b \in \mathbb{C}$ with $u \neq 0$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-s)^m - (n+1)H_m) x^n W_{j+mn} + (-s)^m (n+1) x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}.$$

(c) *If* $(-s)^m x^2 - xH_m + 1 = u(x-c)^2 = 0$ *for some* $u, c \in \mathbb{C}$ *with* $u \neq 0$, *i.e.,* $x = c$, *then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}.$$

Proof.

(a) Note that if the roots of characteristic equation (1.5) are distinct then

$$\begin{aligned}\sum_{k=0}^{n} x^k W_{mk+j} &= x^n W_{mn+j} + \sum_{k=0}^{n-1} x^k W_{mk+j} \\ &= x^n W_{mn+j} + \sum_{k=0}^{n-1} (A_1\alpha^{mk+j} + A_2\beta^{mk+j})x^k \\ &= x^n W_{mn+j} + A_1\alpha^j\left(\frac{(\alpha^m x)^n - 1}{\alpha^m x - 1}\right) + A_2\beta^j\left(\frac{(\beta^m x)^n - 1}{\beta^m x - 1}\right).\end{aligned}$$

Simplifying the last equalities in the last two expression imply (4.1) as required. If the roots of characteristic equation (1.5) are equal then the proof is similar.

(b) We use (4.1). For $x = a$ and $x = b$, the right hand side of the above sum formula 4.1) is an indeterminate form. Now, we can use L'Hospital rule. Then we get (b) by using

$$\begin{aligned}\sum_{k=0}^{n} a^k W_{mk+j} &= \left.\frac{\frac{d}{dx}\left(((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}\right)}{\frac{d}{dx}\left((-s)^m x^2 - xH_m + 1\right)}\right|_{x=a} \\ &= \left.\frac{(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}\right|_{x=a} \\ &= \frac{(a(n+2)(-s)^m - (n+1)H_m)a^n W_{j+mn} + (-s)^m (n+1)a^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m a - H_m}\end{aligned}$$

and

$$\begin{aligned}\sum_{k=0}^{n} b^k W_{mk+j} &= \left.\frac{\frac{d}{dx}\left(((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}\right)}{\frac{d}{dx}\left((-s)^m x^2 - xH_m + 1\right)}\right|_{x=b} \\ &= \left.\frac{(x(n+2)(-s)^m - (n+1)H_m)x^n W_{j+mn} + (-s)^m (n+1)x^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m x - H_m}\right|_{x=b} \\ &= \frac{(b(n+2)(-s)^m - (n+1)H_m)b^n W_{j+mn} + (-s)^m (n+1)b^n W_{mn+j-m} - (-s)^m W_{j-m}}{2(-s)^m b - H_m}.\end{aligned}$$

(c) We use (4.1). For $x = c$, the right hand side of the above sum formula (4.1) is an indeterminate form. Now, we can use L'Hospital rule (twice). Then we get (c) by using

$$\begin{aligned}\sum_{k=0}^{n} c^k W_{mk+j} &= \left.\frac{\frac{d^2}{dx^2}\left(((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + W_j - (-s)^m xW_{j-m}\right)}{\frac{d^2}{dx^2}\left((-1)^m x^2 - xH_m + 1\right)}\right|_{x=c} \\ &= \left.\frac{(n+1)\left((-s)^m (n+2)x^n - nx^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m x^{n-1}W_{mn+j-m}}{2(-s)^m}\right|_{x=c} \\ &= \frac{(n+1)\left((-s)^m (n+2)c^n - nc^{n-1}H_m\right)W_{mn+j} + n(n+1)(-s)^m c^{n-1}W_{mn+j-m}}{2(-s)^m}.\end{aligned}$$

□

Note that (4.1) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-s)^m x - H_m)x^{n+1}W_{mn+j} + (-s)^m x^{n+1}W_{mn+j-m} + x(H_m - (-s)^m x)W_j - (-s)^m xW_{j-m}}{(-s)^m x^2 - xH_m + 1}.$$

4.1 The Case $r = s = 1$: Generalized Fibonacci Numbers

The following theorem presents sum formulas of generalized Fibonacci numbers (the case $r = s = 1$).

Theorem 4.2. *Let x be a real (or complex) number. For all integers m and j, for generalized Fibonacci numbers (the case $r = s = 1$) we have the following sum formulas:*

(a) *If $(-1)^m x^2 - xL_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-1)^m x - L_m)x^{n+1}W_{mn+j} + (-1)^m x^{n+1}W_{mn+j-m} + W_j - (-1)^m xW_{j-m}}{(-1)^m x^2 - xL_m + 1}. \quad (4.2)$$

(b) *If $(-1)^m x^2 - xL_m + 1 = (x-a)(x-b) = 0$ for some $a, b \in \mathbb{C}$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-1)^m - (n+1)L_m)x^n W_{j+mn} + (-1)^m (n+1)x^n W_{mn+j-m} - (-1)^m W_{j-m}}{2(-1)^m x - L_m}.$$

(c) *If $(-1)^m x^2 - xL_m + 1 = (x-c)^2 = 0$ for some $c \in \mathbb{C}$ then*

$$\sum_{k=0}^{n} c^k W_{mk+j} = \frac{(n+1)\left((-1)^m (n+2)c^n - nc^{n-1}L_m\right) W_{mn+j} + n(n+1)(-1)^m c^{n-1} W_{mn+j-m}}{2(-1)^m}.$$

Proof. Take $r = s = 1$ and $H_n = L_n$ in Theorem 4.1.
Note that (4.2) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-1)^m x - L_m)x^{n+1}W_{mn+j} + (-1)^m x^{n+1}W_{mn+j-m} + x(L_m - (-1)^m x)W_j - (-1)^m xW_{j-m}}{(-1)^m x^2 - xL_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 4.1. *For generalized Fibonacci numbers (the case $r = s = 1$) we have the following sum formulas for $n \geq 0$:*

(a) *(The case: $m = 1, j = 0$).*
If $-x^2 - x + 1 \neq 0$, i.e., $x \neq -\frac{1}{2} + \frac{1}{2}\sqrt{5}$, $x \neq -\frac{1}{2} - \frac{1}{2}\sqrt{5}$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(x+1)x^{n+1}W_n + x^{n+1}W_{n-1} - (W_1 - W_0)x - W_0}{x^2 + x - 1},$$

and
if $-x^2 - x + 1 = 0$, i.e., $x = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $x = -\frac{1}{2} - \frac{1}{2}\sqrt{5}$, then

$$\sum_{k=0}^{n} x^k W_k = \frac{(2x + 1 + n(x+1))x^n W_n + (n+1)x^n W_{n-1} - (W_1 - W_0)}{2x + 1}.$$

(b) *(The case: $m = 2, j = 0$).*
If $x^2 - 3x + 1 \neq 0$, i.e., $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}$, $x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5}$, then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(x-3)x^{n+1}W_{2n} + x^{n+1}W_{2n-2} + (W_1 - 2W_0)x + W_0}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0$, i.e., $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ or $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, then

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(2x - 3 + n(x-3))x^n W_{2n} + (n+1)x^n W_{2n-2} + (W_1 - 2W_0)}{2x - 3}.$$

(c) *(The case:* $m=2,\ j=1$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}=\frac{(x-3)x^{n+1}W_{2n+1}+x^{n+1}W_{2n-1}-(W_1-W_0)x+W_1}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{2k+1}=\frac{((x-3)n+2x-3)\,x^nW_{2n+1}+(n+1)x^nW_{2n-1}-(W_1-W_0)}{2x-3}.$$

(d) *(The case:* $m=-1,\ j=0$*).*
If $-x^2+x+1\neq 0$, *i.e.,* $x\neq \frac{1}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-k}=\frac{x^{n+1}W_{-n+1}+(x-1)x^{n+1}W_{-n}-W_1x-W_0}{x^2-x-1},$$

and
if $-x^2+x+1=0$, *i.e.,* $x=\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-k}=\frac{(n+1)x^nW_{-n+1}+(2x-1+n(x-1))x^nW_{-n}-W_1}{2x-1}.$$

(e) *(The case:* $m=-2,\ j=0$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-2k}=\frac{x^{n+1}W_{-2n+2}+(x-3)x^{n+1}W_{-2n}-W_2x+W_0}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-2k}=\frac{(n+1)x^nW_{-2n+2}+(2x-3+n(x-3))x^nW_{-2n}-W_2}{2x-3}.$$

(f) *(The case:* $m=-2,\ j=1$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-2k+1}=\frac{x^{n+1}W_{-2n+3}+(x-3)x^{n+1}W_{-2n+1}-W_3x+W_1}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n}x^kW_{-2k+1}=\frac{(n+1)x^nW_{-2n+3}+(2x-3+n(x-3))x^nW_{-2n+1}-W_3}{2x-3}.$$

From the above proposition, we have the following corollary which gives sum formulas of Fibonacci numbers (take $W_n=F_n$ with $F_0=0,F_1=1$).

Corollary 4.3. *For $n \geq 0,$ Fibonacci numbers have the following properties:*

(a) *(The case: $m = 1, j = 0$).*
If $-x^2 - x + 1 \neq 0,$ i.e., $x \neq -\frac{1}{2} + \frac{1}{2}\sqrt{5}, x \neq -\frac{1}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_k = \frac{(x+1)x^{n+1}F_n + x^{n+1}F_{n-1} - x}{x^2 + x - 1},$$

and
if $-x^2 - x + 1 = 0,$ i.e., $x = -\frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $x = -\frac{1}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_k = \frac{(2x + 1 + n(x+1))x^n F_n + (n+1)x^n F_{n-1} - 1}{2x + 1}.$$

(b) *(The case: $m = 2, j = 0$).*
If $x^2 - 3x + 1 \neq 0,$ i.e., $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{2k} = \frac{(x-3)x^{n+1}F_{2n} + x^{n+1}F_{2n-2} + x}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0,$ i.e., $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ or $x = \frac{3}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{2k} = \frac{(2x - 3 + n(x-3))x^n F_{2n} + (n+1)x^n F_{2n-2} + 1}{2x - 3}.$$

(c) *(The case: $m = 2, j = 1)$).*
If $x^2 - 3x + 1 \neq 0,$ i.e., $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{2k+1} = \frac{(x-3)x^{n+1}F_{2n+1} + x^{n+1}F_{2n-1} - x + 1}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0,$ i.e., $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ or $x = \frac{3}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{2k+1} = \frac{((x-3)n + 2x - 3)\, x^n F_{2n+1} + (n+1)x^n F_{2n-1} - 1}{2x - 3}.$$

(d) *(The case: $m = -1, j = 0$).*
If $-x^2 + x + 1 \neq 0,$ i.e., $x \neq \frac{1}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{1}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{-k} = \frac{x^{n+1}F_{-n+1} + (x-1)x^{n+1}F_{-n} - x}{x^2 - x - 1},$$

and
if $-x^2 + x + 1 = 0,$ i.e., $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ or $x = \frac{1}{2} - \frac{1}{2}\sqrt{5},$ then

$$\sum_{k=0}^{n} x^k F_{-k} = \frac{(n+1)x^n F_{-n+1} + (2x - 1 + n(x-1))x^n F_{-n} - 1}{2x - 1}.$$

(e) *(The case:* $m=-2,\ j=0$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k} = \frac{x^{n+1}F_{-2n+2}+(x-3)x^{n+1}F_{-2n}-x}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k} = \frac{(n+1)x^nF_{-2n+2}+(2x-3+n(x-3))x^nF_{-2n}-1}{2x-3}.$$

(f) *(The case:* $m=-2,\ j=1$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1} = \frac{x^{n+1}F_{-2n+3}+(x-3)x^{n+1}F_{-2n+1}-2x+1}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k F_{-2k+1} = \frac{(n+1)x^nF_{-2n+3}+(2x-3+n(x-3))x^nF_{-2n+1}-2}{2x-3}.$$

Taking $W_n=L_n$ with $L_0=2, L_1=1$ in the last proposition, we have the following corollary which presents sum formulas of Lucas numbers.

Corollary 4.4. *For* $n\geq 0$, *Lucas numbers have the following properties:*

(a) *(The case:* $m=1,\ j=0$*).*
If $-x^2-x+1\neq 0$, *i.e.,* $x\neq -\frac{1}{2}+\frac{1}{2}\sqrt{5},\ x\neq -\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_k = \frac{(x+1)x^{n+1}L_n+x^{n+1}L_{n-1}+x-2}{x^2+x-1},$$

and
if $-x^2-x+1=0$, *i.e.,* $x=-\frac{1}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=-\frac{1}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_k = \frac{(2x+1+n(x+1))x^nL_n+(n+1)x^nL_{n-1}+1}{2x+1}.$$

(b) *(The case:* $m=2,\ j=0$*).*
If $x^2-3x+1\neq 0$, *i.e.,* $x\neq \frac{3}{2}+\frac{1}{2}\sqrt{5},\ x\neq \frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k} = \frac{(x-3)x^{n+1}L_{2n}+x^{n+1}L_{2n-2}+(L_1-2L_0)x+2}{x^2-3x+1},$$

and
if $x^2-3x+1=0$, *i.e.,* $x=\frac{3}{2}+\frac{1}{2}\sqrt{5}$ *or* $x=\frac{3}{2}-\frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k} = \frac{(2x-3+n(x-3))x^nL_{2n}+(n+1)x^nL_{2n-2}-3}{2x-3}.$$

(c) *(The case:* $m = 2, j = 1$*).*
If $x^2 - 3x + 1 \neq 0$, *i.e.,* $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k+1} = \frac{(x-3)x^{n+1}L_{2n+1} + x^{n+1}L_{2n-1} + x + 1}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0$, *i.e.,* $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ *or* $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{2k+1} = \frac{((x-3)n + 2x - 3)\, x^n L_{2n+1} + (n+1)x^n L_{2n-1} + 1}{2x - 3}.$$

(d) *(The case:* $m = -1, j = 0$*).*
If $-x^2 + x + 1 \neq 0$, *i.e.,* $x \neq \frac{1}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{1}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-k} = \frac{x^{n+1}L_{-n+1} + (x-1)x^{n+1}L_{-n} - x - 2}{x^2 - x - 1},$$

and
if $-x^2 + x + 1 = 0$, *i.e.,* $x = \frac{1}{2} + \frac{1}{2}\sqrt{5}$ *or* $x = \frac{1}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-k} = \frac{(n+1)x^n L_{-n+1} + (2x - 1 + n(x-1))x^n L_{-n} - 1}{2x - 1}.$$

(e) *(The case:* $m = -2, j = 0$*).*
If $x^2 - 3x + 1 \neq 0$, *i.e.,* $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k} = \frac{x^{n+1}L_{-2n+2} + (x-3)x^{n+1}L_{-2n} - 3x + 2}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0$, *i.e.,* $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ *or* $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k} = \frac{(n+1)x^n L_{-2n+2} + (2x - 3 + n(x-3))x^n L_{-2n} - 3}{2x - 3}.$$

(f) *(The case:* $m = -2, j = 1$*).*
If $x^2 - 3x + 1 \neq 0$, *i.e.,* $x \neq \frac{3}{2} + \frac{1}{2}\sqrt{5}, x \neq \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1} = \frac{x^{n+1}L_{-2n+3} + (x-3)x^{n+1}L_{-2n+1} - 4x + 1}{x^2 - 3x + 1},$$

and
if $x^2 - 3x + 1 = 0$, *i.e.,* $x = \frac{3}{2} + \frac{1}{2}\sqrt{5}$ *or* $x = \frac{3}{2} - \frac{1}{2}\sqrt{5}$, *then*

$$\sum_{k=0}^{n} x^k L_{-2k+1} = \frac{(n+1)x^n L_{-2n+3} + (2x - 3 + n(x-3))x^n L_{-2n+1} - 4}{2x - 3}.$$

Taking $x = 1$ in the last two corollaries we get the following corollary.

Corollary 4.5. *For* $n \geq 0$, *Fibonacci numbers, and Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} F_k = 2F_n + F_{n-1} - 1.$
- **(b)** $\sum_{k=0}^{n} F_{2k} = 2F_{2n} - F_{2n-2} - 1.$
- **(c)** $\sum_{k=0}^{n} F_{2k+1} = 2F_{2n+1} - F_{2n-1}.$
- **(d)** $\sum_{k=0}^{n} F_{-k} = -F_{-n+1} + 1.$
- **(e)** $\sum_{k=0}^{n} F_{-2k} = -F_{-2n+2} + 2F_{-2n} + 1.$
- **(f)** $\sum_{k=0}^{n} F_{-2k+1} = -F_{-2n+3} + 2F_{-2n+1} + 1.$

2.

- **(a)** $\sum_{k=0}^{n} L_k = 2L_n + L_{n-1} - 1.$
- **(b)** $\sum_{k=0}^{n} L_{2k} = 2L_{2n} - L_{2n-2} + 1.$
- **(c)** $\sum_{k=0}^{n} L_{2k+1} = 2L_{2n+1} - L_{2n-1} - 2.$
- **(d)** $\sum_{k=0}^{n} L_{-k} = -L_{-n+1} + 3.$
- **(e)** $\sum_{k=0}^{n} L_{-2k} = -L_{-2n+2} + 2L_{-2n} + 1.$
- **(f)** $\sum_{k=0}^{n} L_{-2k+1} = -L_{-2n+3} + 2L_{-2n+1} + 3.$

4.2 The Case $r = 2, s = 1$: Generalized Pell Numbers

The following theorem presents sum formulas of generalized Pell numbers (the case $r = 2, s = 1$).

Theorem 4.6. *Let x be a real (or complex) number. For all integers m and j, for generalized Pell numbers we have the following sum formulas:*

(a) *if $(-1)^m x^2 - xQ_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-1)^m x - Q_m)x^{n+1}W_{mn+j} + (-1)^m x^{n+1}W_{mn+j-m} + W_j - (-1)^m xW_{j-m}}{(-1)^m x^2 - xQ_m + 1}. \tag{4.3}$$

(b) *If $(-1)^m x^2 - xQ_m + 1 = (x-a)(x-b) = 0$ for some $a, b \in \mathbb{C}$ and $a \neq b$, i.e., $x = a$ or $x = b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-1)^m - (n+1)Q_m)x^n W_{j+mn} + (-1)^m (n+1)x^n W_{mn+j-m} - (-1)^m W_{j-m}}{2(-1)^m x - Q_m}.$$

(c) *If $(-1)^m x^2 - xQ_m + 1 = (x-c)^2 = 0$ for some $c \in \mathbb{C}$ then*

$$\sum_{k=0}^{n} c^k W_{mk+j} = \frac{(n+1)\left((-1)^m (n+2)c^n - nc^{n-1}Q_m\right)W_{mn+j} + n(n+1)(-1)^m c^{n-1}W_{mn+j-m}}{2(-1)^m}.$$

Proof. *Take* $r = 2, s = 1$ and $H_n = Q_n$ in Theorem 4.1. □

Note that (4.3) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-1)^m x - Q_m)x^{n+1}W_{mn+j} + (-1)^m x^{n+1}W_{mn+j-m} + x(Q_m - (-1)^m x)W_j - (-1)^m xW_{j-m}}{(-1)^m x^2 - xQ_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 4.2. *For generalized Pell numbers (the case $r = 2, s = 1$) we have the following sum formulas for $n \geq 0$:*

(a) *(The case:* $m = 1,\, j = 0$*).*
If $-x^2 - 2x + 1 \neq 0$, *i.e.,* $x \neq -1 + \sqrt{2}$, $x \neq -1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(x+2)x^{n+1}W_n + x^{n+1}W_{n-1} - (W_1 - 2W_0)x - W_0}{x^2 + 2x - 1},$$

and
if $-x^2 - 2x + 1 = 0$, *i.e.,* $x = -1 + \sqrt{2}$ *or* $x = -1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(2x + 2 + n(2+x))x^n W_n + (n+1)x^n W_{n-1} - (W_1 - 2W_0)}{2x + 2}.$$

(b) *(The case:* $m = 2,\, j = 0$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(x-6)x^{n+1}W_{2n} + x^{n+1}W_{2n-2} + (2W_1 - 5W_0)x + W_0}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(2x - 6 + n(x-6))x^n W_{2n} + (n+1)x^n W_{2n-2} + (2W_1 - 5W_0)}{2x - 6}.$$

(c) *(The case:* $m = 2,\, j = 1$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(x-6)x^{n+1}W_{2n+1} + x^{n+1}W_{2n-1} - (W_1 - 2W_0)x + W_1}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{\left((x-6)n + 2(x-3)\right)x^n W_{2n+1} + (n+1)x^n W_{2n-1} - (W_1 - 2W_0)}{2(x-3)}.$$

(d) *(The case:* $m = -1,\, j = 0$*).*
If $-x^2 + 2x + 1 \neq 0$, *i.e.,* $x \neq 1 + \sqrt{2}$, $x \neq 1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1} + (x-2)x^{n+1}W_{-n} - W_1 x - W_0}{x^2 - 2x - 1},$$

and
if $-x^2 + 2x + 1 = 0$, *i.e.,* $x = 1 + \sqrt{2}$ *or* $x = 1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(n+1)x^n W_{-n+1} + (2x - 2 + n(x-2))x^n W_{-n} - W_1}{2x - 2}.$$

(e) *(The case:* $m = -2,\, j = 0$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{x^{n+1}W_{-2n+2} + (x-6)x^{n+1}W_{-2n} - W_2 x + W_0}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{(n+1)x^n W_{-2n+2} + (2x - 6 + n(x-6))x^n W_{-2n} - W_2}{2x-6}.$$

(f) *(The case:* $m = -2$, $j = 1$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{x^{n+1} W_{-2n+3} + (x-6)x^{n+1} W_{-2n+1} - W_3 x + W_1}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{(n+1)x^n W_{-2n+3} + (2x - 6 + n(x-6))x^n W_{-2n+1} - W_3}{2x-6}.$$

From the above proposition, we have the following corollary which gives sum formulas of Pell numbers (take $W_n = P_n$ with $P_0 = 0, P_1 = 1$).

Corollary 4.7. *For* $n \geq 0$, *Pell numbers have the following properties:*

(a) *(The case:* $m = 1$, $j = 0$*).*
If $-x^2 - 2x + 1 \neq 0$, *i.e.,* $x \neq -1 + \sqrt{2}$, $x \neq -1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_k = \frac{(x+2)x^{n+1} P_n + x^{n+1} P_{n-1} - x}{x^2 + 2x - 1},$$

and
if $-x^2 - 2x + 1 = 0$, *i.e.,* $x = -1 + \sqrt{2}$ *or* $x = -1 - \sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_k = \frac{(2x + 2 + n(2+x))x^n P_n + (n+1)x^n P_{n-1} - 1}{2x+2}.$$

(b) *(The case:* $m = 2$, $j = 0$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k} = \frac{(x-6)x^{n+1} P_{2n} + x^{n+1} P_{2n-2} + 2x}{x^2 - 6x + 1},$$

and
if $x^2 - 6x + 1 = 0$, *i.e.,* $x = 3 + 2\sqrt{2}$ *or* $x = 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k} = \frac{(2x - 6 + n(x-6))x^n P_{2n} + (n+1)x^n P_{2n-2} + 2}{2x-6}.$$

(c) *(The case:* $m = 2$, $j = 1$*).*
If $x^2 - 6x + 1 \neq 0$, *i.e.,* $x \neq 3 + 2\sqrt{2}$, $x \neq 3 - 2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k+1} = \frac{(x-6)x^{n+1} P_{2n+1} + x^{n+1} P_{2n-1} - x + 1}{x^2 - 6x + 1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{2k+1} = \frac{((x-6)n+2(x-3))\,x^n P_{2n+1}+(n+1)x^n P_{2n-1}-1}{2(x-3)}.$$

(d) *(The case:* $m=-1, j=0$*).*
If $-x^2+2x+1\neq 0$, *i.e.,* $x\neq 1+\sqrt{2}$, $x\neq 1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-k} = \frac{x^{n+1}P_{-n+1}+(x-2)x^{n+1}P_{-n}-x}{x^2-2x-1},$$

and
if $-x^2+2x+1=0$, *i.e.,* $x=1+\sqrt{2}$ *or* $x=1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-k} = \frac{(n+1)x^n P_{-n+1}+(2x-2+n(x-2))x^n P_{-n}-1}{2x-2}.$$

(e) *(The case:* $m=-2, j=0$*).*
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k} = \frac{x^{n+1}P_{-2n+2}+(x-6)x^{n+1}P_{-2n}-2x}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k} = \frac{(n+1)x^n P_{-2n+2}+(2x-6+n(x-6))x^n P_{-2n}-2}{2x-6}.$$

(f) *(The case:* $m=-2, j=1$*).*
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1} = \frac{x^{n+1}P_{-2n+3}+(x-6)x^{n+1}P_{-2n+1}-5x+1}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k P_{-2k+1} = \frac{(n+1)x^n P_{-2n+3}+(2x-6+n(x-6))x^n P_{-2n+1}-5}{2x-6}.$$

Taking $W_n=Q_n$ with $Q_0=2, Q_1=2$ in the last proposition, we have the following corollary which presents sum formulas of Pell-Lucas numbers.

Corollary 4.8. *For* $n\geq 0$, *Pell-Lucas numbers have the following properties:*

(a) *(The case:* $m=1, j=0$*).*
If $-x^2-2x+1\neq 0$, *i.e.,* $x\neq -1+\sqrt{2}$, $x\neq -1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_k = \frac{(x+2)x^{n+1}Q_n+x^{n+1}Q_{n-1}+2x-2}{x^2+2x-1},$$

and
if $-x^2-2x+1=0$, *i.e.,* $x=-1+\sqrt{2}$ *or* $x=-1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_k = \frac{(2x+2+n(2+x))x^n Q_n + (n+1)x^n Q_{n-1} + 2}{2x+2}.$$

(b) *(The case:* $m=2, j=0$*).*
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}, x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k} = \frac{(x-6)x^{n+1}Q_{2n} + x^{n+1}Q_{2n-2} - 6x + 2}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k} = \frac{(2x-6+n(x-6))x^n Q_{2n} + (n+1)x^n Q_{2n-2} - 6}{2x-6}.$$

(c) *(The case:* $m=2, j=1$*).*
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}, x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k+1} = \frac{(x-6)x^{n+1}Q_{2n+1} + x^{n+1}Q_{2n-1} + 2x + 2}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{2k+1} = \frac{((x-6)n + 2(x-3))\, x^n Q_{2n+1} + (n+1)x^n Q_{2n-1} + 2}{2(x-3)}.$$

(d) *(The case:* $m=-1, j=0$*).*
If $-x^2+2x+1\neq 0$, *i.e.,* $x\neq 1+\sqrt{2}, x\neq 1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-k} = \frac{x^{n+1}Q_{-n+1} + (x-2)x^{n+1}Q_{-n} - 2x - 2}{x^2-2x-1},$$

and
if $-x^2+2x+1=0$, *i.e.,* $x=1+\sqrt{2}$ *or* $x=1-\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-k} = \frac{(n+1)x^n Q_{-n+1} + (2x-2+n(x-2))x^n Q_{-n} - 2}{2x-2}.$$

(e) *(The case:* $(m=-2, j=0$*).*
If $x^2-6x+1\neq 0$, *i.e.,* $x\neq 3+2\sqrt{2}, x\neq 3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k} = \frac{x^{n+1}Q_{-2n+2} + (x-6)x^{n+1}Q_{-2n} - 6x + 2}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, *i.e.,* $x=3+2\sqrt{2}$ *or* $x=3-2\sqrt{2}$, *then*

$$\sum_{k=0}^{n} x^k Q_{-2k} = \frac{(n+1)x^n Q_{-2n+2} + (2x-6+n(x-6))x^n Q_{-2n} - 6}{2x-6}.$$

(f) *(The case: $m=-2$, $j=1$).*
If $x^2-6x+1\neq 0$, i.e., $x\neq 3+2\sqrt{2}$, $x\neq 3-2\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_{-2k+1} = \frac{x^{n+1}Q_{-2n+3}+(x-6)x^{n+1}Q_{-2n+1}-14x+2}{x^2-6x+1},$$

and
if $x^2-6x+1=0$, i.e., $x=3+2\sqrt{2}$ or $x=3-2\sqrt{2}$, then

$$\sum_{k=0}^{n} x^k Q_{-2k+1} = \frac{(n+1)x^n Q_{-2n+3}+(2x-6+n(x-6))x^n Q_{-2n+1}-14}{2x-6}.$$

Taking $x=1$ in the last two corollaries we get the following corollary.

Corollary 4.9. *For $n\geq 0$, Pell numbers and Pell-Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} P_k = \frac{1}{2}(3P_n+P_{n-1}-1)$.
- **(b)** $\sum_{k=0}^{n} P_{2k} = \frac{1}{4}(5P_{2n}-P_{2n-2}-2)$.
- **(c)** $\sum_{k=0}^{n} P_{2k+1} = \frac{1}{4}(5P_{2n+1}-P_{2n-1})$.
- **(d)** $\sum_{k=0}^{n} P_{-k} = \frac{1}{2}(-P_{-n+1}+P_{-n}+1)$.
- **(e)** $\sum_{k=0}^{n} P_{-2k} = \frac{1}{4}(-P_{-2n+2}+5P_{-2n}+2)$.
- **(f)** $\sum_{k=0}^{n} P_{-2k+1} = \frac{1}{4}(-P_{-2n+3}+5P_{-2n+1}+4)$.

2.

- **(a)** $\sum_{k=0}^{n} Q_k = \frac{1}{2}(3Q_n+Q_{n-1})$.
- **(b)** $\sum_{k=0}^{n} Q_{2k} = \frac{1}{4}(5Q_{2n}-Q_{2n-2}+4)$.
- **(c)** $\sum_{k=0}^{n} Q_{2k+1} = \frac{1}{4}(5Q_{2n+1}-Q_{2n-1}-4)$.
- **(d)** $\sum_{k=0}^{n} Q_{-k} = \frac{1}{2}(-Q_{-n+1}+Q_{-n}+4)$.
- **(e)** $\sum_{k=0}^{n} Q_{-2k} = \frac{1}{4}(-Q_{-2n+2}+5Q_{-2n}+4)$.
- **(f)** $\sum_{k=0}^{n} Q_{-2k+1} = \frac{1}{4}(-Q_{-2n+3}+5Q_{-2n+1}+12)$.

4.3 The Case $r=1, s=2$: Generalized Jacobsthal Numbers

The following theorem presents sum formulas of generalized Jacobsthal numbers (the case $r=1, s=2$).

Theorem 4.10. *Let x be a real (or complex) number. For all integers m and j, for generalized Jacobsthal numbers we have the following sum formulas:*

(a) *if $(-2)^m x^2 - xj_m + 1 \neq 0$ then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{((-2)^m x - j_m)x^{n+1}W_{mn+j}+(-2)^m x^{n+1}W_{mn+j-m}+W_j-(-2)^m xW_{j-m}}{(-2)^m x^2 - xj_m+1}. \tag{4.4}$$

(b) *If $(-2)^m x^2 - xj_m + 1 = (x-a)(x-b)=0$ for some $a,b\in\mathbb{C}$ and $a\neq b$, i.e., $x=a$ or $x=b$, then*

$$\sum_{k=0}^{n} x^k W_{mk+j} = \frac{(x(n+2)(-2)^m-(n+1)j_m)x^n W_{j+mn}+(-2)^m(n+1)x^n W_{mn+j-m}-(-2)^m W_{j-m}}{2(-2)^m x - j_m}.$$

(c) *If* $(-2)^m x^2 - xj_m + 1 = (x-c)^2 = 0$ *for some* $c \in \mathbb{C}$ *then*

$$\sum_{k=0}^{n} c^k W_{mk+j} = \frac{(n+1)\left((-2)^m (n+2)c^n - nc^{n-1}L_m\right) W_{mn+j} + n(n+1)(-2)^m c^{n-1} W_{mn+j-m}}{2(-2)^m}.$$

Proof. *Take* $r = 1, s = 2$ and $H_n = j_n$ in Theorem 4.1.
Note that (4.4) can be written in the following form

$$\sum_{k=1}^{n} x^k W_{mk+j} = \frac{((-2)^m x - j_m)x^{n+1} W_{mn+j} + (-2)^m x^{n+1} W_{mn+j-m} + x(j_m - (-2)^m x)W_j - (-2)^m x W_{j-m}}{(-2)^m x^2 - xj_m + 1}.$$

As special cases of m and j in the last Theorem, we obtain the following proposition.

Proposition 4.3. *For generalized Jacobsthal numbers (the case* $r = 1, s = 2$*) we have the following sum formulas:*

(a) *(The case:* $m = 1, j = 0$*).*
If $-2x^2 - x + 1 \neq 0$, *i.e.,* $x \neq -1, x \neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(2x+1)x^{n+1} W_n + 2x^{n+1} W_{n-1} - (W_1 - W_0)x - W_0}{2x^2 + x - 1},$$

and
if $-2x^2 - x + 1 = 0$, *i.e.,* $x = -1$ *or* $x = \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k W_k = \frac{(4x + 1 + n(2x+1))x^n W_n + 2(n+1)x^n W_{n-1} - (W_1 - W_0)}{4x+1}.$$

(b) *(The case:* $m = 2, j = 0$*).*
If $4x^2 - 5x + 1 \neq 0$, *i.e.,* $x \neq 1, x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(4x-5)x^{n+1} W_{2n} + 4x^{n+1} W_{2n-2} + (W_1 - 3W_0)x + W_0}{4x^2 - 5x + 1},$$

and
if $4x^2 - 5x + 1 = 0$, *i.e.,* $x = 1$ *or* $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k} = \frac{(4x(n+2) - 5(n+1))x^n W_{2n} + 4(n+1)x^n W_{2n-2} + (W_1 - 3W_0)}{8x-5}.$$

(c) *(The case:* $m = 2, j = 1$*).*
If $4x^2 - 5x + 1 \neq 0$, *i.e.,* $x \neq 1, x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{(4x-5)x^{n+1} W_{2n+1} + 4x^{n+1} W_{2n-1} - 2(W_1 - W_0)x + W_1}{4x^2 - 5x + 1},$$

and
if $4x^2 - 5x + 1 = 0$, *i.e.,* $x = 1$ *or* $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k W_{2k+1} = \frac{((4x-5)n + 8x - 5)x^n W_{2n+1} + 4(n+1)x^n W_{2n-1} - 2(W_1 - W_0)}{8x-5}.$$

(d) *(The case: $m=-1$, $j=0$).*

If $x^2-x-2\neq 0$, i.e., $x\neq 2$, $x\neq -1$, then

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{x^{n+1}W_{-n+1}+(x-1)x^{n+1}W_{-n}-W_1x-2W_0}{x^2-x-2},$$

and

if $x^2-x-2=0$, i.e., $x=2$ or $x=-1$, then

$$\sum_{k=0}^{n} x^k W_{-k} = \frac{(n+1)x^n W_{-n+1}+(2x-1+n(x-1))x^n W_{-n}-W_1}{2x-1}.$$

(e) *(The case: $m=-2$, $j=0$).*

If $x^2-5x+4\neq 0$, i.e., $x\neq 1$, $x\neq 4$, then

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{x^{n+1}W_{-2n+2}+(x-5)x^{n+1}W_{-2n}-W_2x+4W_0}{x^2-5x+4},$$

and

if $x^2-5x+4=0$, i.e., $x=1$ or $x=4$, then

$$\sum_{k=0}^{n} x^k W_{-2k} = \frac{(n+1)x^n W_{-2n+2}+(2x-5+n(x-5))x^n W_{-2n}-W_2}{2x-5}.$$

(f) *(The case: $m=-2$, $j=1$).*

If $x^2-5x+4\neq 0$, i.e., $x\neq 1$, $x\neq 4$, then

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{x^{n+1}W_{-2n+3}+(x-5)x^{n+1}W_{-2n+1}-W_3x+4W_1}{x^2-5x+4},$$

and

if $x^2-5x+4=0$, i.e., $x=1$ or $x=4$, then

$$\sum_{k=0}^{n} x^k W_{-2k+1} = \frac{(n+1)x^n W_{-2n+3}+(2x-5+n(x-5))x^n W_{-2n+1}-W_3}{2x-5}.$$

From the above proposition, we have the following corollary which gives sum formulas of Jacobsthal numbers (take $W_n=J_n$ with $J_0=0, J_1=1$).

Corollary 4.11. *For $n\geq 0$, Jacobsthal numbers have the following properties:*

(a) *(The case: $m=1$, $j=0$).*

If $-2x^2-x+1\neq 0$, i.e., $x\neq -1$, $x\neq \frac{1}{2}$, then

$$\sum_{k=0}^{n} x^k J_k = \frac{(2x+1)x^{n+1}J_n+2x^{n+1}J_{n-1}-x}{2x^2+x-1},$$

and

if $-2x^2-x+1=0$, i.e., $x=-1$ or $x=\frac{1}{2}$, then

$$\sum_{k=0}^{n} x^k J_k = \frac{(4x+1+n(2x+1))x^n J_n+2(n+1)x^n J_{n-1}-1}{4x+1}.$$

(b) *(The case:* $m = 2,\ j = 0$*).*
If $4x^2 - 5x + 1 \neq 0$, ***i.e.,*** $x \neq 1,\ x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k J_{2k} = \frac{(4x-5)x^{n+1}J_{2n} + 4x^{n+1}J_{2n-2} + x}{4x^2 - 5x + 1},$$

and
if $4x^2 - 5x + 1 = 0$, ***i.e.,*** $x = 1$ *or* $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k J_{2k} = \frac{(4x(n+2) - 5(n+1))x^n J_{2n} + 4(n+1)x^n J_{2n-2} + 1}{8x - 5}.$$

(c) *(The case:* $m = 2,\ j = 1$*).*
If $4x^2 - 5x + 1 \neq 0$, ***i.e.,*** $x \neq 1,\ x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k J_{2k+1} = \frac{(4x-5)x^{n+1}J_{2n+1} + 4x^{n+1}J_{2n-1} - 2x + 1}{4x^2 - 5x + 1},$$

and
if $4x^2 - 5x + 1 = 0$, ***i.e.,*** $x = 1$ *or* $x = \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k J_{2k+1} = \frac{((4x-5)n + 8x - 5)x^n J_{2n+1} + 4(n+1)x^n J_{2n-1} - 2}{8x - 5}.$$

(d) *(The case:* $m = -1,\ j = 0$*).*
If $x^2 - x - 2 \neq 0$, ***i.e.,*** $x \neq 2,\ x \neq -1$, *then*

$$\sum_{k=0}^{n} x^k J_{-k} = \frac{x^{n+1}J_{-n+1} + (x-1)x^{n+1}J_{-n} - x}{x^2 - x - 2},$$

and
if $x^2 - x - 2 = 0$, ***i.e.,*** $x = 2$ *or* $x = -1$, *then*

$$\sum_{k=0}^{n} x^k J_{-k} = \frac{(n+1)x^n J_{-n+1} + (2x - 1 + n(x-1))x^n J_{-n} - 1}{2x - 1}.$$

(e) *(The case:* $m = -2,\ j = 0$*).*
If $x^2 - 5x + 4 \neq 0$, ***i.e.,*** $x \neq 1,\ x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k J_{-2k} = \frac{x^{n+1}J_{-2n+2} + (x-5)x^{n+1}J_{-2n} - x}{x^2 - 5x + 4},$$

and
if $x^2 - 5x + 4 = 0$, ***i.e.,*** $x = 1$ *or* $x = 4$, *then*

$$\sum_{k=0}^{n} x^k J_{-2k} = \frac{(n+1)x^n J_{-2n+2} + (2x - 5 + n(x-5))x^n J_{-2n} - 1}{2x - 5}.$$

(f) *(The case:* $m = -2,\ j = 1$*).*
If $x^2 - 5x + 4 \neq 0$, ***i.e.,*** $x \neq 1,\ x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k J_{-2k+1} = \frac{x^{n+1}J_{-2n+3} + (x-5)x^{n+1}J_{-2n+1} - 3x + 4}{x^2 - 5x + 4},$$

and
if $x^2-5x+4=0$, *i.e.,* $x=1$ *or* $x=4$, *then*

$$\sum_{k=0}^{n} x^k J_{-2k+1} = \frac{(n+1)x^n J_{-2n+3} + (2x-5+n(x-5))x^n J_{-2n+1} - 3}{2x-5}.$$

Taking $W_n = j_n$ with $j_0 = 2, j_1 = 1$ in the last proposition, we have the following corollary which presents sum formulas of Jacobsthal-Lucas numbers.

Corollary 4.12. *For* $n \geq 0$, *Jacobsthal-Lucas numbers have the following properties:*

(a) *(The case:* $m=1, j=0$*).*
If $-2x^2-x+1 \neq 0$, *i.e.,* $x \neq -1$, $x \neq \frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k j_k = \frac{(2x+1)x^{n+1} j_n + 2x^{n+1} j_{n-1} + x - 2}{2x^2+x-1},$$

and
if $-2x^2-x+1=0$, *i.e.,* $x=-1$ *or* $x=\frac{1}{2}$, *then*

$$\sum_{k=0}^{n} x^k j_k = \frac{(4x+1+n(2x+1))x^n j_n + 2(n+1)x^n j_{n-1} + 1}{4x+1}.$$

(b) *(The case:* $m=2, j=0$*).*
If $4x^2-5x+1 \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k j_{2k} = \frac{(4x-5)x^{n+1} j_{2n} + 4x^{n+1} j_{2n-2} - 5x + 2}{4x^2-5x+1},$$

and
if $4x^2-5x+1=0$, *i.e.,* $x=1$ *or* $x=\frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k j_{2k} = \frac{(4x(n+2)-5(n+1))x^n j_{2n} + 4(n+1)x^n j_{2n-2} - 5}{8x-5}.$$

(c) *(The case:* $m=2, j=1$*).*
If $4x^2-5x+1 \neq 0$, *i.e.,* $x \neq 1$, $x \neq \frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k j_{2k+1} = \frac{(4x-5)x^{n+1} j_{2n+1} + 4x^{n+1} j_{2n-1} + 2x + 1}{4x^2-5x+1},$$

and
if $4x^2-5x+1=0$, *i.e.,* $x=1$ *or* $x=\frac{1}{4}$, *then*

$$\sum_{k=0}^{n} x^k j_{2k+1} = \frac{((4x-5)n+8x-5)x^n j_{2n+1} + 4(n+1)x^n j_{2n-1} + 2}{8x-5}.$$

(d) *(The case:* $m=-1, j=0$*).*
If $x^2-x-2 \neq 0$, *i.e.,* $x \neq 2$, $x \neq -1$, *then*

$$\sum_{k=0}^{n} x^k j_{-k} = \frac{x^{n+1} j_{-n+1} + (x-1)x^{n+1} j_{-n} - x - 4}{x^2-x-2},$$

and
if $x^2 - x - 2 = 0$, *i.e.,* $x = 2$ *or* $x = -1$, *then*

$$\sum_{k=0}^{n} x^k j_{-k} = \frac{(n+1)x^n j_{-n+1} + (2x - 1 + n(x-1))x^n j_{-n} - 1}{2x - 1}.$$

(e) *(The case:* $m = -2, j = 0$*).*
If $x^2 - 5x + 4 \neq 0$, *i.e.,* $x \neq 1, x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k} = \frac{x^{n+1} j_{-2n+2} + (x-5)x^{n+1} j_{-2n} - 5x + 8}{x^2 - 5x + 4},$$

and
if $x^2 - 5x + 4 = 0$, *i.e.,* $x = 1$ *or* $x = 4$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k} = \frac{(n+1)x^n j_{-2n+2} + (2x - 5 + n(x-5))x^n j_{-2n} - 5}{2x - 5}.$$

(f) *(The case:* $m = -2, j = 1$*).*
If $x^2 - 5x + 4 \neq 0$, *i.e.,* $x \neq 1, x \neq 4$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1} = \frac{x^{n+1} j_{-2n+3} + (x-5)x^{n+1} j_{-2n+1} - 7x + 4}{x^2 - 5x + 4},$$

and
if $x^2 - 5x + 4 = 0$, *i.e.,* $x = 1$ *or* $x = 4$, *then*

$$\sum_{k=0}^{n} x^k j_{-2k+1} = \frac{(n+1)x^n j_{-2n+3} + (2x - 5 + n(x-5))x^n j_{-2n+1} - 7}{2x - 5}.$$

Taking $x = 1$ in the last two corollaries we get the following corollary.

Corollary 4.13. *For* $n \geq 0$, *Jacobsthal numbers and Jacobsthal-Lucas numbers have the following properties:*

1.

- **(a)** $\sum_{k=0}^{n} J_k = \frac{1}{2}(3J_n + 2J_{n-1} - 1)$.
- **(b)** $\sum_{k=0}^{n} J_{2k} = \frac{1}{3}((3-n)J_{2n} + 4(n+1)J_{2n-2} + 1)$.
- **(c)** $\sum_{k=0}^{n} J_{2k+1} = \frac{1}{3}((3-n)J_{2n+1} + 4(n+1)J_{2n-1} - 2)$.
- **(d)** $\sum_{k=0}^{n} J_{-k} = \frac{1}{2}(-J_{-n+1} + 1)$.
- **(e)** $\sum_{k=0}^{n} J_{-2k} = \frac{1}{3}(-(n+1)J_{-2n+2} + (4n+3)J_{-2n} + 1)$.
- **(f)** $\sum_{k=0}^{n} J_{-2k+1} = \frac{1}{3}(-(n+1)J_{-2n+3} + (4n+3)J_{-2n+1} + 3)$.

2.

- **(a)** $\sum_{k=0}^{n} j_k = \frac{1}{2}(3j_n + 2j_{n-1} - 1)$.
- **(b)** $\sum_{k=0}^{n} j_{2k} = \frac{1}{3}((3-n)j_{2n} + 4(n+1)j_{2n-2} - 5)$.
- **(c)** $\sum_{k=0}^{n} j_{2k+1} = \frac{1}{3}((3-n)j_{2n+1} + 4(n+1)j_{2n-1} + 2)$.
- **(d)** $\sum_{k=0}^{n} j_{-k} = \frac{1}{2}(-j_{-n+1} + 5)$.
- **(e)** $\sum_{k=0}^{n} j_{-2k} = \frac{1}{3}(-(n+1)j_{-2n+2} + (4n+3)j_{-2n} + 5)$.
- **(f)** $\sum_{k=0}^{n} j_{-2k+1} = \frac{1}{3}(-(n+1)j_{-2n+3} + (4n+3)j_{-2n+1} + 7)$.

5 CONCLUSION

Recently, there have been so many studies of the sequences of numbers in the literature and the sequences of numbers were widely used in many research areas, such as architecture, nature, art, physics and engineering. In this work, sum identities were proved. The method used in this paper can be used for the other linear recurrence sequences, too. We have written sum identities in terms of the generalized Fibonacci sequence, and then we have presented the formulas as special cases the corresponding identity for the Fibonacci, Fibonacci-Lucas, Pell, Pell-Lucas, Jacobsthal and Jacobsthal-Lucas sequences. All the listed identities in the propositions and corollaries may be proved by induction, but that method of proof gives no clue about their discovery. We give the proofs to indicate how these identities, in general, were discovered. Furthermore, some identities and recurrence properties of generalized Fibonacci sequence were studied.

We can mention some applications of sum formulas. Computations of the Frobenius norm, spectral norm, maximum column length norm and maximum row length norm of circulant (r-circulant, geometric circulant, semicirculant) matrices with the generalized m-step Fibonacci sequences require the sum of the numbers of the sequences.

COMPETING INTERESTS

Author has declared that no competing interests exist.

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