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Incidence Matrix of a Semigraph

C. M. Deshpande[1], Yogeshri Gaidhani[2*] and B. P. Athawale[1]

[1]*Department of Mathematics, College of Engineering, Pune, India.*
[2]*Department of Mathematics, M.E.S. Abasaheb Garware College, Pune, India.*

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Abstract

Semigraph was defined by Sampathkumar as a generalization of graph. In this paper incidence matrix which represents semigraph uniquely and characterization of such a matrix is obtained. Some properties of incidence matrix of semigraph are studied. Structure of incidence matrix of some special classes of semigraphs are obtained.

Keywords: Semigraph; Incidence matrix; i-semigraphical matrix.

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1 Introduction

Semigraph was introduced by Sampathkumar [1] as a generalization of a graph. Various basic concepts are defined by Sampathkumar and semigraphs have been investigated by many authors on the lines of graphs. Kamath and S. Hebbar [2] have worked primirarily on domination theory. They have studied domination critical semigraphs. Venkatkrishnan et al. [3] have studied theory of bipartite semigraphs. They have introduced the concept of hyperdomination in bipartite semigraphs.

Some matrices associated with semigraphs were defined by Sampathkumar, none of these matrices considered individually represents a semigraph uniquely. In fact various other definitions of matrix representations were given later, but no unique representation. The authors [4] have defined

*Corresponding author: E-mail: yogeshrigaidhani@gmail.com

adjacency matrix of semigraph which gives unique representation of a semigraph which does not contain a particular forbidden structure.

We address this question in this paper by defining incidence matrix of a semigraph which provides a unique matrix representation of a semigraph. When graph is considered as a special case of semigraph, our definition of incidence matrix coincides with the definition of incidence matrix of a graph.

In graph theory incidence matrix and adjacency matrix of a graph give unique matrix representations of a discrete structure graph. They are effectively used for storing graphs [5]. Hence this definition will facilitate use of a semigraph as a data structure.

1.1 Basic definitions

Here we define some basic terms needed to read this paper. For the terms not defined here, the reader may refer to [6, 7]. Semigraph was introduced by Sampathkumar [1] as a generalization of graph.

Definition 1.1. [1] *Semigraph G* is an ordered pair of two sets V and X, where V is a non-empty set whose elements are called vertices of G and X is a set of n-tuples, called edges of G, of distinct vertices, for various n (n at least 2) satisfying the following conditions:

1. (SG1) - Any two edges have at most one vertex in common.
2. (SG2) - Two edges $(u_1, u_2, \cdots, u_n)$ and $(v_1, v_2, \cdots, v_m)$ are considered to be equal if
   (a) m = n and
   (b) either $u_i = v_i$ for $i = 1, 2, \cdots, n$ or $u_i = v_{n+1-i}$, for $i = 1, 2, \cdots, n$.

Thus the edge $(u_1, u_2, \cdots, u_m)$ is same as $(u_m, u_{m-1}, \cdots, u_1)$.
For the edge $e = (u_1, u_2, \cdots, u_n)$, u_1 and u_n are called the *end* vertices of e and $u_2, u_3, \cdots, u_{n-1}$ are called the *middle* vertices of e.

Example 1.1. *Consider a semigraph $G = (V, X)$ in Fig. 1 where $V = \{v_1, v_2, \cdots, v_9\}$ and $X = \{(v_1, v_2, v_3, v_4), (v_4, v_5), (v_1, v_6, v_5), (v_4, v_6, v_7), (v_5, v_7), (v_2, v_6), (v_5, v_9)\}$.*

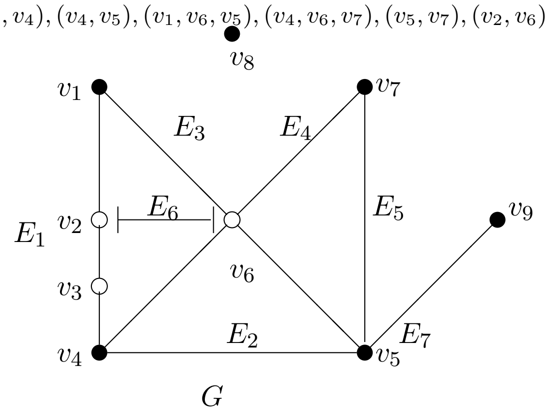

Fig.1. Example-1

In G, vertices v_1, v_4, v_5, v_7 and v_9 are the end vertices; v_3 is a middle vertex; v_2 and v_6 are the middle-end vertices and v_8 is an isolated vertex.

There are more than one types of degrees that are defined for a vertex of a semigraph and all of them can be obtained from our definition of incidence matrix.

1. ***Degree,*** $deg(v)$, is the number of edges containing v.
2. ***Middle degree,*** $d_m(v)$, is the number of edges containing v as a middle vertex and ***end degree,*** $d_e(v)$, is the number of edges containing v as an end vertex.
3. ***Adjacent degree,*** $d_a(v)$, is the number of vertices adjacent to v, where two vertices are said to be adjacent if they belong to an edge.
4. ***Consecutive adjacent degree,*** $d_{ca}(v)$, is the number of vertices consecutively adjacent to v, where two vertices are said to be consecutively adjacent if they belong to an edge and appear consecutively in the ordered $n-$tuple.
5. ***me-degree***, $d_{me}(v)$ is an ordered pair $(d_m(v), d_e(v))$.

2 Incidence Matrix of a Semigraph

Definition 2.1. Let $G(V, X)$ be a semigraph with vertex set $V = \{1, 2, \cdots, p\}$ and edge set $X = \{e_1, e_2, \cdots, e_q\}$ where $e_j = (i_1, i_2, \cdots, i_{k_j})$.
For each $j = 1, 2, \cdots, q$ define

$$a_{ij} = \begin{cases} 0; & if \;\; i \notin \{i_1, i_2, \cdots, i_{k_j}\} \\ 1; & if \;\; i = i_1, i_2 \; and \; k_j = 2 \\ r; & if \; i = i_r, \;\; for \; r = 1, 2, 3, \cdots, k_j \; and \; k_j \geq 3 \end{cases}$$

$I_S(G) = [a_{ij}]$ is a $p \times q$ matrix called the **incidence matrix** associated with the semigraph G.

Example 2.1. *If $G = (V, X)$ is a semigraph with $V = \{1, 2, 3, 4, 5, 6\}$,*
$X = \{e_1 = (1, 5, 2, 6), e_2 = (3, 5, 4), e_3 = (2, 3)\}$
Incidence matrix of G is

$$I_S(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \end{array} \begin{array}{c} \begin{array}{ccc} e_1 & e_2 & e_3 \end{array} \\ \begin{pmatrix} 1 & 0 & 0 \\ 3 & 0 & 1 \\ 0 & 1 & 1 \\ 0 & 3 & 0 \\ 2 & 2 & 0 \\ 4 & 0 & 0 \end{pmatrix} \end{array}$$

We observe that this matrix representation of a semigraph is unique.

Definition 2.2. A $p \times q$ matrix M is said to be $i - semigraphical$ if there exists a semigraph G on p vertices with q edges such that $I_S(G) = M$.

Following theorem gives necessary and sufficient conditions for a matrix of order $p \times q$ to be i-semigraphical:

Theorem 2.2. *A $p \times q$ matrix $A = [a_{ij}]$ is $i - semigraphcal$ if and only if it satisfies the following conditions:*

1. $a_{i,j} \in \{0, 1, 2, \cdots, p\} \quad \forall i, j$
2. *Every column of A contains precisely one or two entries equal to 1.*
   (a) *If j^{th} column contains two entries equal to 1 then all the remaining entries in that column are equal to zero.*

(b) *If j^{th} column contains one entry equal to 1, then the number of nonzero entries in that column is at least three and all the nonzero entries of that column form the sequence $1, 2, \cdots, k_j$, where k_j is the largest entry in j^{th} column. Each entry of this sequence appears exactly once.*

3. *Any 2×2 submatrix of A has at least one entry equal to zero.*

Proof: If A is $p \times q$ matrix which is i-semigraphical then by definition of incidence matrix, it satisfies the above conditions.

Conversely suppose that A is a $p \times q$ matrix satisfying the above conditions. We label the rows and columns as $1, 2, \cdots, p$ and $e_1, e_2, \cdots, e_q$ respectively.
Define $V = \{1, 2, \cdots, p\}$ and $X = \{e_1, e_2, \cdots, e_q\}$, where $e_j = (i_1, i_2, \cdots, i_s)$ and $i_1, i_2, \cdots, i_s$ are obtained from conditions $2(a)$ and $2(b)$.
Claim: $G = (V, X)$ is a semigraph.

From condition 1, every edge is tuple of elements of V. By condition 2, any edge has precisely two end vertices. If condition 3 is not satisfied then there exists a 2×2 submatrix of A such that all the entries of this submatrix are nonzero. Let the submatrix be generated by rows i_1, i_2 and the columns j_1, j_2. Then the edges e_{j_1}, e_{j_2} have two common vertices i_1, i_2. A contradiction. Hence, condition 3 guarantees that no two edges have more than one vertex in common. Condition $2(b)$ defines the order of vertices in any edge. Hence G is a semigraph such that incidence matrix of G, $I_S(G) = A$.

2.1 Some Properties of incidence matrix $I_s(G)$

Let $\xi > 1$ denote the maximum entry in a column. ξ will appear in a column if and only if the corresponding edge e has cardinality at least 3. Also note that ξ has different values for different columns.

1. Number of non zero entries in j^{th} column gives the cardinality of the edge e_j.
2. Number of entries in i^{th} row which are equal to 1 or ξ gives end degree of the vertex i.
3. Number of nonzero entries in i^{th} row which are not equal to 1 *or* ξ gives middle degree of the vertex i.
4. From (2) and (3) above we can find degree of a vertex $d(v) = d_e(v) + d_m(v)$ and also write the (m, e)-degree sequence of G
5. A row with all zeros represents an isolated vertex.
6. If G is a graph then $I_S(G)$ coincides with incidence matrix of G.
7. Interchanging of any two columns (/rows) corresponds to re-labeling of edges (/vertices)and hence gives an isomorphic semigraph.
8. As the elements of $I_S(G)$ can be considered to be elements the finite field $\mathcal{F}_n$, where n is the number of vertices of G, this matrix defines a linear code [8].

3 Structure of Incidence Matrix of Some Special Classes of Semigraphs

3.1 Weakly connected semigraphs

Definition 3.1. [1] A $v_0 - v_n$ ***walk*** in a semigraph G is a sequence of vertices $v_0\ v_1 \cdots v_n$, such that v_i, v_{i+1} are adjacent vertices for $i = 0$ to $n - 1$. A $v_0 - v_n$ walk is said to be a ***strong walk*** if the vertices v_i and v_{i+1} are consecutively adjacent for $i = 0$ to $n - 1$. Otherwise it is said to be a ***weak walk***.

A semigraph $G(V, X)$ is said to be weakly connected if and only if between any two vertices u and v in V, there exists a weak walk in G.

Theorem 3.1. *$G(V, X)$ is a disconnected semigraph with two components B_1 and B_2 if and only if* $I_S(G) = \begin{pmatrix} B_1(G) & Z \\ Z & B_2(G) \end{pmatrix}$,

where $B_i(G)$ is the incidence matrix associated with component B_i, $i = 1, 2$, and Z is a zero matrix of appropriate order.

Proof: If G is a semigraph with all isolated vertices then proof of the result is trivial.
Let $G(V, X)$be a semigraph with $V = \{1, 2, 3, \cdots, p\}$ and edge set $X \neq \phi$.
Let G be a disconnected semigraph with two components B_1 and B_2.
Let $V_1 = \{1\}$. Consider any edge containing vertex 1. Add all the vertices of this edge to V_1. Repeat this procedure for all the edges containing vertex 1.
Therefore $V_1 = \{$Set of all the vertices which are adjacent to vertex $1\} \bigcup \{1\}$.
Let i be a vertex in V_1, $i \neq 1$. Add all the vertices of G which are adjacent to i to V_1. Continue this process till V_1 is closed with respect to this operation.
i.e. if $i \in V_1$ and j is adjacent to i then $j \in V_1$.
Let $V_2 = V - V_1$. Then $V_2 \neq \phi$ because G is disconnected.
Without loss of generality, $V_1 = V(B_1)$ and $V_2 = V(B_2)$.
$V_1 = \{i_1, i_2, \cdots, i_r\}$ and $V_2 = \{i_{r+1}, i_{r+2}, \cdots, i_p\}$
Order the vertices of V as $\{i_1, i_2, \cdots, i_r, i_{r+1}, i_{r+2}, \cdots, i_p\}$
$X(B_1) = \{e_1, e_2, \cdots, e_n\}$ and $X(B_2) = \{e_{n+1}, e_{n+2}, \cdots, e_q\}$. Let $B_1(G)$ and $B_2(G)$ be the incidence matrices corresponding to B_1 and B_2 respectively. $B_1(G)$ is of order $r \times n$ and $B_2(G)$ is of order $p - r \times q - n$.
For every $j = 1, 2, \cdots, n, e_j$ does not contain $i_{r+1}, i_{r+2}, \cdots, i_p$.
For every $j = n + 1, n + 2, \cdots, q, e_j$ does not contain $i_1, i_2, \cdots, i_r$.
Hence, $I[i, j] = 0$; for $j = 1, 2, \cdots, n$ and $i = r + 1, r + 2, \cdots, q$
and $I[i, j] = 0$; for $j = n + 1, n + 2, \cdots, q$ and $i = 1, 2, \cdots, r$.

Hence, $I_S(G) = \begin{pmatrix} B_1(G) & Z \\ Z & B_2(G) \end{pmatrix}$.

Conversely, suppose that incidence matrix of a semigraph G is in the given form. We observe that a vertex in B_1 is not adjacent to any vertex in B_2. Hence G is disconnected with two components B_1 and B_2.

□

The above result can be extended to a semigaph with n components.

Corollary 3.2. *G is a semigraph with n components if and only if*

$$I_S(G) = \begin{pmatrix} B_1(G) & \cdots & Z \\ \vdots & \ddots & \vdots \\ Z & \cdots & B_n(G) \end{pmatrix}$$

3.2 Star semigraphs

Definition 3.2. A semigraph $G(V, X)$ is called a star semigraph if G has exactly one vertex common to all the edges.

Let $|V| = p$ and $|X| = q$. Let the common vertex be i_1. Further without loss of generality suppose that i_1 is an end vertex of $e_1, e_2, \cdots, e_k$ and a middle vertex of $e_{k+1}, \cdots, e_q$.

Hence, $a_{i_1 j} = 1$ *or* e, if $1 \leq j \leq k$ and $a_{i_1 j}$ is nonzero entry not equal to $1, \xi$ if $k+1 \leq j \leq q$.

In particular, we consider two cases.

Case 1: i_1, is an end vertex of all the edges of G i.e. $k = q$.

Then, in $I_S(G)$,

1. There is precisely one row, the i_1^{th} row, such that $a_{i_1 j} = 1,\ or\ \xi\ \ \forall j$.
2. As G is connected, every row of $I_S(G)$ contains at least one nonzero entry.
3. Since every vertex $i \neq i_1$ belongs to exactly one edge, exactly one entry in i^{th} row is nonzero. Further this nonzero entry is 1 *or* ξ for exactly q rows, while the remaining rows contain entry from $\{1, 2, \cdots, p\}$. Also number of rows containing the entry i_1 indicate the number of edges of G with cardinality ≥ 2. If there are r such semigraphical edges then exactly r numbers are missing from $\{2, 3, \cdots, p\}$ in the matrix. Lastly any number from $\{2, 3, \cdots, p\}$ appears atmost once in $I_S(G)$.

Case 2: i_1, is a middle vertex of all the edges i.e. $k = 0$

Then, in $I_S(G)$,

1. There is precisely one row, the i_1^{th} row, such that $a_{i_1 j} \neq 1, 0,\ or\ \xi\ \ \forall j$.
2. As G is connected, every row of $I_S(G)$ contains at least one nonzero entry.
3. If i is any row, $i \neq i_1$, then by condition 3 of theorem 2.1 i^{th} row contains exactly one nonzero entry.
4. $d(i_1) = q$ and hence there are precisely q rows containing exactly one entry equal to 1 and precisely q rows containing exactly one entry equal to ξ. All other $p - 2q - 1$ rows contain exactly one nonzero entry from $\{1, 2, \cdots, p\}$.

3.3 k-uniform cycle semigraphs

Definition 3.3. (1) A semigraph G is called a $k-uniform$ semigraph if for edge e of G, $\ |e| = k$.

Definition 3.4. (1) A ***path*** is a walk in a semigraph in which all the vertices are distinct.

A ***cycle*** is a closed path.

If $G(V, X)$ is a $k - uniform\ cycle$ semigraph with $V = \{1, 2 \cdots, p\}$ and $X = \{e_1, e_2, \cdots e_q\}$, then it is easy to observe that $p > q$.

Let G be a k-uniform cycle semigraph.

1. G has q vertices with $me-$degree $(0, 2)$, say $\{1, 2 \cdots, q\}$ and $p - q$ vertices with $me-$degree $(1, 0)$, say $\{q + 1, q + 2, \cdots, p\}$.
2. Every edge of G contains 2 end vertices and $k - 2$ middle vertices. Hence, $p = (k - 1)q$.
3. Every column of $I_S(G)$ contains k nonzero entries equal to $\{1, 2, \cdots, k\ (or\ \xi)\}$.
4. Every row of $I_S(G)$ contains precisely one or two nonzero entries.
   - (a) $\forall i \in \{1, 2, \cdots, q\}$, $\ i^{th}$ row contains two nonzero entries equal to 1 *or* ξ.
   - (b) $\forall i \in \{q + 1, q + 2, \cdots, p\}$, $\ i^{th}$ row contains exactly one nonzero entry not equal to 1 *or* ξ.

3.4 Dual semigraph of a hamiltonian graph

Bhave et al. have characterized the degree sequence of a dual semigraph of a hamiltonian graph. We characterize the incidence matrix of such a semigraph.

Definition 3.5. [9] Let G be a (p,q) graph with the degree of every vertex at least two. ***Dual semigraph*** of G is a semigraph $G^*(q,p)$ where the vertices of G^* are the edges of G and the edge e_i^* of G^*, $i = \{1, 2, \cdots, p\}$ is the set of all edges in G that are incident with the vertex v_i with some order.

Definition 3.6. [9] A Semigraph G^* is called the ***dual semigraph of a Hamiltonian graph (DSHG)*** if there exists a Hamiltonian graph G with cycle $C = \{e_1, e_2, \cdots, e_p\}$ such that its dual semigraph is G^* and the end vertices of edges of G^* belong to C.

Definition 3.7. [9] A sequence $\pi = \{(m_1, e_1), (m_2, e_2), \cdots, (m_q, e_q)\}$ is called ***potentially DSHG*** if there exists a realization of π which is DSHG.

Bhave et al. [9] proved that a semigraph $G(V,X)$ with degree sequence $\{(0,2)^t, (2,0)^{p-t}\}$, where $t \le p \le \frac{t(t-1)}{2}$, is a dual semigraph of a Hamiltonian graph.
Let $V = \{1, 2, \cdots, p\}$ with $1, 2, \cdots, t$ having (m,e) degree $(0,2)$ and $t+1, t+2, \cdots, p$ having (m,e) degree $(2,0)$.
Let $I_S(G)$ be the incidence matrix associated with G. Clearly order of $I_S(G)$ is $p \times t$. For rows $1, 2, \cdots, t$ each row contains exactly two entries as 1 or ξ. Each row i, $i = t+1, \cdots, p$, contains exactly two entries which are nonzero and not equal to 1 and ξ. All the remaining entries of $I_S(G)$ are zero.

Of the t edges in G, at most $min\{(p-t), t\}$ edges can contain middle vertices. Let $e_{i_1}, e_{i_2}, \cdots, e_{i_k}$ be the edges containing at least one middle vertex. Remaining $t-k$ edges, if any, are graphical edges. Graphical edges exist only if $p < 2t$.
If necessary by swapping the edges we get that $I_S(G)$ in the following form.

$$I_S(G) = \begin{pmatrix} Block \;\; containing \;\; (1, \; \xi) \; and \; 0 & \vdots & Graphical \; edges \\ \cdots\cdots\cdots\cdots\cdots\cdots & \vdots & \cdots\cdots\cdots \\ Block \;\; without \; 1 \; and \; \xi & \vdots & \text{Z} \end{pmatrix}$$

If $I_S(G)$ does not contain any graphical edges then

$$I_S(G) = \begin{pmatrix} Block \;\; containing \;\; (1, \; \xi) \; and \; 0 \\ \cdots\cdots\cdots\cdots\cdots\cdots \\ Block \;\; without 1, \; \xi \end{pmatrix}$$

Example 3.3. *Consider (m,e) degree sequence $d = \{(0,2)^4, (2,0)^1\}$.*

Let $\{1, 2, 3, 4\}$ be the vertices having (m,e) degree (0, 2) and $\{5\}$ be the vertex having (m,e) degree (2,0). d is degree sequence of dual semigraph of a Hamiltonian graph.
$V = \{1, 2, 3, 4, 5\}$, $X = \{e_1 = (1,2), e_2 = (3,4), e_3 = (1,5,3), e_4 = (2,5,4)\}$
$G(V,X)$ has (m,e) degree sequence d. G is a semigraphical realization of d.

$$I_S(G) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ 5 \end{array} \begin{array}{c} \begin{array}{cccc} e_1 & e_2 & e_3 & e_4 \end{array} \\ \begin{pmatrix} 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 0 & 1 & 3 & 0 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 2 & 2 \end{pmatrix} \end{array}$$

By interchanging columns of $I_S(G)$,

$$I_S(G_1) = \begin{array}{c} \\ 1 \\ 2 \\ 3 \\ 4 \\ \\ 5 \end{array} \begin{array}{c} \begin{array}{ccccc} e_3 & e_4 & & e_1 & e_2 \end{array} \\ \begin{pmatrix} 1 & 0 & \vdots & 1 & 0 \\ 0 & 1 & \vdots & 1 & 0 \\ \xi & 0 & \vdots & 0 & 1 \\ 0 & \xi & \vdots & 0 & 1 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 2 & 2 & \vdots & 0 & 0 \end{pmatrix} \end{array}$$

Note that G and G_1 are same semigraphs with edges relabeled.

4 Conclusions

a In the section (1.1), we define basic terms needed to read this paper.

b In the subsection (2), we define incidence matrix of a semigraph. Necessary and sufficient conditions for a matrix to be i-semigraphical is proved.

c In section (2.1), some properties of incidence matrix of semigraph are studied.

d In section (3.1), necessary and sufficient condition for a semigraph to be weakly connected are obtained.

e In section (3.2) and (3.3), properties of incidence matrix of Star semigraph and k-uniform semigraphs are studied.

f In section (3.4), structure of dual semigraph of a hamiltonian graph is obtained.

Competing Interests

The authors declare that no competing interests exist.

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