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The Convolution Sums $\sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m)$

Aeran Kim[1*]

[1] *1599-6-bungi, Keumam-2-dong, Deokjin-gu, Jeonju-si, Jeollabuk-do, 561-806, South Korea.*

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Abstract

In this paper, we calculate the convolution sum formulae of

$$\sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m)$$

for $n(\in\mathbb{N})$ and an odd positive integer e and f. Moreover we obtain some identities induced from $\sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m)$ and deduce a coefficients relation.

Keywords: Divisor functions; Convolution sums

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**Corresponding author: E-mail: ae_ran_kim@hotmail.com*

1 Introduction

For $n \in \mathbb{N}$, $s \in \mathbb{N} \cup \{0\}$, $q \in \mathbb{C}$ with $|q| < 1$, we define the important divisor function and the infinite product sums, which also appear in many areas of number theory:

$$\sigma_s(n) = \sum_{d|n} d^s, \qquad \Delta(q) := \sum_{n=1}^{\infty} \tau(n) q^n = q \prod_{n=1}^{\infty} (1-q^n)^{24},$$
$$A(q) := \sum_{n=1}^{\infty} a(n) q^n = \Delta(q^2)^{\frac{1}{2}} = q \prod_{n=1}^{\infty} (1-q^{2n})^{12},$$
$$B(q) := \sum_{n=1}^{\infty} b(n) q^n = \left(\Delta(q)\Delta(q^2)\right)^{\frac{1}{3}} = q \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^8,$$
$$C(q) := \sum_{n=1}^{\infty} c(n) q^n = \left(\Delta(q)^4\Delta(q^2)\right)^{\frac{1}{6}} = q \prod_{n=1}^{\infty} (1-q^n)^{16} (1-q^{2n})^4, \tag{1.1}$$
$$D(q) := \sum_{n=1}^{\infty} d(n) q^n = \left(\Delta(q)^2\Delta(q^2)\Delta(q^4)^2\right)^{\frac{1}{6}} = q^2 \prod_{n=1}^{\infty} (1-q^n)^8 (1-q^{2n})^4 (1-q^{4n})^8,$$
$$E(q) := \sum_{n=1}^{\infty} e(n) q^n = \left(\Delta(q^2)\Delta(q^4)^4\right)^{\frac{1}{6}} = q^3 \prod_{n=1}^{\infty} (1-q^{2n})^4 (1-q^{4n})^{16},$$
$$F(q) := \sum_{n=1}^{\infty} f(n) q^n = \left(\frac{\Delta(q^4)^4}{\Delta(q)}\right)^{\frac{1}{3}} = q^5 \prod_{n=1}^{\infty} (1-q^n)^{24} (1+q^n)^{32} (1+q^{2n})^{32}.$$

Here we obtain the relation between $c(n)$ and $d(n)$:

Theorem 1.1. *Let $n \in \mathbb{N}$. Then we have*

$$e(n) = -\frac{1}{8}\left\{d(n) - 32d(\frac{n}{2}) - c(\frac{n}{2})\right\}.$$

Theorem 1.1 enables us to express $e(n)$ with $c(n)$ and $d(n)$. Let $q \in \mathbb{C}$ be such that $|q| < 1$. The Eisenstein series $L(q)$, $M(q)$, and $N(q)$ are

$$L(q) = 1 - 24\sum_{n=1}^{\infty} \sigma_1(n) q^n, \tag{1.2}$$

$$M(q) = 1 + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.3}$$

$$N(q) = 1 - 504\sum_{n=1}^{\infty} \sigma_5(n) q^n \tag{1.4}$$

by ([1], p. 318). Lahiri ([2], p. 149) has derived the following identities from the work of Ramanujan [3] :

$$L^2(q) = 1 - 288\sum_{n=1}^{\infty} n\sigma_1(n) q^n + 240\sum_{n=1}^{\infty} \sigma_3(n) q^n, \tag{1.5}$$

$$M^2(q) = 1 + 480\sum_{n=1}^{\infty} \sigma_7(n) q^n, \tag{1.6}$$

$$M^3(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{432000}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \quad (1.7)$$

$$L(q)M(q) = 1 + 720\sum_{n=1}^{\infty}n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n, \quad (1.8)$$

$$L^2(q)M(q) = 1 + 1728\sum_{n=1}^{\infty}n^2\sigma_3(n)q^n - 2016\sum_{n=1}^{\infty}n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n, \quad (1.9)$$

$$L(q)M^2(q) = 1 + 720\sum_{n=1}^{\infty}n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n, \quad (1.10)$$

$$N^2(q) = 1 + \frac{65520}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n - \frac{762048}{691}\sum_{n=1}^{\infty}\tau(n)q^n, \quad (1.11)$$

$$L(q)N(q) = 1 - 1008\sum_{n=1}^{\infty}n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n. \quad (1.12)$$

For $e, f, m, n \in \mathbb{N}$ we define

$$I_{m,e,f}(n) := \sum_{m=1}^{n-1} m\sigma_e(m)\sigma_f(n-m) \quad \text{and} \quad I_{e,f}(n) := \sum_{m=1}^{n-1}\sigma_e(m)\sigma_f(n-m).$$

Ramanujan [3] and Lahiri [2], [4] have shown that $I_{e,f}$ can be expressed as :

$$\begin{aligned}
I_{1,1}(n) &= \frac{5}{12}\sigma_3(n) + \frac{(1-6n)}{12}\sigma_1(n),\\
I_{1,3}(n) &= \frac{7}{80}\sigma_5(n) + \frac{(1-3n)}{24}\sigma_3(n) - \frac{1}{240}\sigma_1(n),\\
I_{1,5}(n) &= \frac{5}{126}\sigma_7(n) + \frac{(1-2n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(n),\\
I_{3,3}(n) &= \frac{1}{120}\sigma_7(n) - \frac{1}{120}\sigma_3(n),\\
I_{1,7}(n) &= \frac{11}{480}\sigma_9(n) + \frac{(2-3n)}{48}\sigma_7(n) - \frac{1}{480}\sigma_1(n),\\
I_{3,5}(n) &= \frac{11}{5040}\sigma_9(n) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(n),\\
I_{1,9}(n) &= \frac{455}{30404}\sigma_{11}(n) + \frac{(5-6n)}{120}\sigma_9(n) + \frac{1}{264}\sigma_1(n) - \frac{36}{3455}\tau(n),\\
I_{3,7}(n) &= \frac{91}{110560}\sigma_{11}(n) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(n) + \frac{15}{2764}\tau(n),\\
I_{5,5}(n) &= \frac{65}{174132}\sigma_{11}(n) + \frac{1}{252}\sigma_5(n) - \frac{3}{691}\tau(n),\\
I_{1,11}(n) &= \frac{691}{65520}\sigma_{13}(n) + \frac{(1-n)}{24}\sigma_{11}(n) - \frac{691}{65520}\tau(n),\\
I_{3,9}(n) &= \frac{1}{2640}\sigma_{13}(n) - \frac{1}{240}\sigma_9(n) + \frac{1}{264}\sigma_3(n),\\
I_{5,7}(n) &= \frac{1}{10080}\sigma_{13}(n) + \frac{1}{504}\sigma_7(n) - \frac{1}{480}\sigma_5(n).
\end{aligned}$$

And we have already obtained :

Proposition 1.1. *Let $n \in \mathbb{N}$. Then we have*

(a) *(See ([5], p. 155))*

$$I_{m,1,1}(n) = \frac{1}{24} n \left\{ 5\sigma_3(n) - (6n-1)\,\sigma_1(n) \right\},$$

(b) *(See ([5], p. 157))*

$$I_{m,1,3}(n) = \frac{7}{240} n\sigma_5(n) - \frac{1}{40} n^2 \sigma_3(n) - \frac{1}{240} n\sigma_1(n),$$
$$I_{m,3,1}(n) = \frac{7}{120} n\sigma_5(n) + \left(\frac{1}{24} n - \frac{1}{10} n^2 \right) \sigma_3(n),$$

(c) *(See ([6], Theorem 3.3, Theorem 3.1), ([5], p. 155))*

$$I_{m,1,5}(n) = \frac{1}{504} n \left\{ 5\sigma_7(n) - 6n\sigma_5(n) + \sigma_1(n) \right\},$$
$$I_{m,3,3}(n) = \frac{1}{240} n \left\{ \sigma_7(n) - \sigma_3(n) \right\},$$
$$I_{m,5,1}(n) = \frac{1}{168} n \left\{ 5\sigma_7(n) - (12n-7)\,\sigma_5(n) \right\},$$

(d) *(See ([6], Theorem 3.8 (a), (b), Theorem 3.7, Theorem 3.5))*

$$I_{m,1,7}(n) = \frac{1}{7200} \left\{ 33n\sigma_9(n) - 50n^2\sigma_7(n) - 15n\sigma_1(n) + 32\tau(n) \right\},$$
$$I_{m,3,5}(n) = \frac{1}{12600} \left\{ 11n\sigma_9(n) + 25n\sigma_3(n) - 36\tau(n) \right\},$$
$$I_{m,5,3}(n) = \frac{1}{8400} \left\{ 11n\sigma_9(n) - 35n\sigma_5(n) + 24\tau(n) \right\},$$
$$I_{m,7,1}(n) = \frac{1}{1800} \left\{ 33n\sigma_9(n) - 25n\,(4n-3)\,\sigma_7(n) - 8\tau(n) \right\}.$$

Similarly, for $e, f, m, n \in \mathbb{N}$ we set

$$T_{m,e,f}(n) := \sum_{m < \frac{n}{2}} m\sigma_e(m)\sigma_f(n-2m),$$
$$T_{e,f}(n) := \sum_{m < \frac{n}{2}} \sigma_e(m)\sigma_f(n-2m),$$
$$U_{m,e,f}(n) := \sum_{m < \frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m),$$
$$U_{e,f}(n) := \sum_{m < \frac{n}{4}} \sigma_e(m)\sigma_f(n-4m).$$

For example, in ([7], p. 45-54) we can see that

$$
\begin{aligned}
T_{1,1}(n) &= \frac{1}{12}\sigma_3(n) + \frac{1}{3}\sigma_3(\frac{n}{2}) + \frac{(1-3n)}{24}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{2}),\\
T_{1,3}(n) &= \frac{1}{48}\sigma_5(n) + \frac{1}{15}\sigma_5(\frac{n}{2}) + \frac{(2-3n)}{48}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{2}),\\
T_{1,5}(n) &= \frac{1}{102}\sigma_7(n) + \frac{32}{1071}\sigma_7(\frac{n}{2}) + \frac{(1-n)}{24}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{2}) - \frac{1}{102}b(n),\\
T_{3,3}(n) &= \frac{1}{2040}\sigma_7(n) + \frac{2}{255}\sigma_7(\frac{n}{2}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{2}) + \frac{1}{272}b(n),\\
T_{5,1}(n) &= \frac{1}{2142}\sigma_7(n) + \frac{2}{51}\sigma_7(\frac{n}{2}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_1(n) - \frac{1}{408}b(n),\\
T_{1,7}(n) &= \frac{17}{2976}\sigma_9(n) + \frac{8}{465}\sigma_9(\frac{n}{2}) + \frac{(4-3n)}{96}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{2})\\
&\quad - \frac{1}{62}c(n) - \frac{16}{31}d(n),\\
T_{3,5}(n) &= \frac{1}{7440}\sigma_9(n) + \frac{4}{1953}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{2})\\
&\quad + \frac{1}{248}c(n) + \frac{4}{31}d(n),\\
T_{5,3}(n) &= \frac{1}{31248}\sigma_9(n) + \frac{1}{465}\sigma_9(\frac{n}{2}) - \frac{1}{240}\sigma_5(\frac{n}{2}) + \frac{1}{504}\sigma_3(n)\\
&\quad - \frac{1}{496}c(n) - \frac{2}{31}d(n),\\
T_{3,7}(n) &= \frac{17}{331680}\sigma_{11}(n) + \frac{8}{10365}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(n) - \frac{1}{480}\sigma_3(\frac{n}{2})\\
&\quad + \frac{91}{22112}\tau(n) + \frac{368}{691}\tau(\frac{n}{2}),\\
T_{7,3}(n) &= \frac{1}{331680}\sigma_{11}(n) + \frac{17}{20730}\sigma_{11}(\frac{n}{2}) - \frac{1}{240}\sigma_7(\frac{n}{2}) - \frac{1}{480}\sigma_3(n)\\
&\quad + \frac{23}{11056}\tau(n) + \frac{91}{1382}\tau(\frac{n}{2}).
\end{aligned}
$$

Also we can find

$$
T_{m,1,1}(n) = \frac{1}{48}n\sigma_3(n) - \frac{1}{48}n^2\sigma_1(n) + \frac{1}{12}n\sigma_3(\frac{n}{2}) + \left(\frac{1}{48}n - \frac{1}{12}n^2\right)\sigma_1(\frac{n}{2}) \tag{1.13}
$$

in ([8], Theorem 4.1) and we can see that

Proposition 1.2. *(See ([9], Theorem 1.1)) Let $n \in \mathbb{N}$. Then we have*

(a)

$$
\begin{aligned}
T_{m,1,3}(n) &= \frac{1}{1440}\left[n\left\{5\sigma_5(n) + 16\sigma_5(\frac{n}{2}) - 9n\sigma_3(n) - 3\sigma_1(\frac{n}{2})\right\} + 4b(n)\right],\\
T_{m,3,1}(n) &= \frac{1}{720}\left[n\left\{\sigma_5(n) + 20\sigma_5(\frac{n}{2}) - (36n-15)\,\sigma_3(\frac{n}{2})\right\} - b(n)\right],
\end{aligned}
$$

(b)

$$
\begin{aligned}
T_{m,1,5}(n) &= \frac{1}{17136}\left\{21n\sigma_7(n) + 64n\sigma_7(\frac{n}{2}) - 51n^2\sigma_5(n) + 17n\sigma_1(\frac{n}{2}) + 1632d(n)\right.\\
&\quad \left.+51c(n) - 21nb(n)\right\},\\
T_{m,3,3}(n) &= \frac{1}{16320}\left\{2n\sigma_7(n) + 32n\sigma_7(\frac{n}{2}) - 34n\sigma_3(\frac{n}{2}) - 544d(n) - 17c(n)\right.\\
&\quad \left.+15nb(n)\right\},\\
T_{m,5,1}(n) &= \frac{1}{22848}\left\{4n\sigma_7(n) + 336n\sigma_7(\frac{n}{2}) - 816n^2\sigma_5(\frac{n}{2}) + 476n\sigma_5(\frac{n}{2}) + 544d(n)\right.\\
&\quad \left.+17c(n) - 21nb(n)\right\}.
\end{aligned}
$$

Furthermore, in ([7], p. 45–54) we can observe that

$$
\begin{aligned}
U_{1,1}(n) &= \frac{1}{48}\sigma_3(n) + \frac{1}{16}\sigma_3(\frac{n}{2}) + \frac{1}{3}\sigma_3(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_1(n) + \frac{(1-6n)}{24}\sigma_1(\frac{n}{4}),\\
U_{1,3}(n) &= \frac{1}{192}\sigma_5(n) + \frac{1}{64}\sigma_5(\frac{n}{2}) + \frac{1}{15}\sigma_5(\frac{n}{4}) + \frac{(4-3n)}{96}\sigma_3(n) - \frac{1}{240}\sigma_1(\frac{n}{4}) - \frac{1}{64}a(n),\\
U_{3,1}(n) &= \frac{1}{3840}\sigma_5(n) + \frac{1}{256}\sigma_5(\frac{n}{2}) + \frac{1}{12}\sigma_5(\frac{n}{4}) + \frac{(1-3n)}{24}\sigma_3(\frac{n}{4}) - \frac{1}{240}\sigma_1(n)\\
&\quad + \frac{1}{256}a(n),\\
U_{1,5}(n) &= \frac{1}{408}\sigma_7(n) + \frac{1}{136}\sigma_7(\frac{n}{2}) + \frac{32}{1071}\sigma_7(\frac{n}{4}) + \frac{(2-n)}{48}\sigma_5(n) + \frac{1}{504}\sigma_1(\frac{n}{4})\\
&\quad - \frac{19}{816}b(n) - \frac{26}{51}b(\frac{n}{2}),\\
U_{3,3}(n) &= \frac{1}{32640}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{255}\sigma_7(\frac{n}{4}) - \frac{1}{240}\sigma_3(n) - \frac{1}{240}\sigma_3(\frac{n}{4})\\
&\quad + \frac{9}{2176}b(n) + \frac{9}{136}b(\frac{n}{2}),\\
U_{5,1}(n) &= \frac{1}{137088}\sigma_7(n) + \frac{1}{2176}\sigma_7(\frac{n}{2}) + \frac{2}{51}\sigma_7(\frac{n}{4}) + \frac{(1-2n)}{24}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_1(n)\\
&\quad - \frac{13}{6528}b(n) - \frac{19}{816}b(\frac{n}{2}),\\
U_{1,7}(n) &= \frac{17}{11904}\sigma_9(n) + \frac{17}{3968}\sigma_9(\frac{n}{2}) + \frac{8}{465}\sigma_9(\frac{n}{4}) + \frac{(8-3n)}{192}\sigma_7(n) - \frac{1}{480}\sigma_1(\frac{n}{4})\\
&\quad - \frac{109}{3968}c(n) - \frac{625}{248}d(n) - \frac{1058}{31}e(n),\\
U_{3,5}(n) &= \frac{1}{119040}\sigma_9(n) + \frac{1}{7936}\sigma_9(\frac{n}{2}) + \frac{4}{1953}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(n) + \frac{1}{504}\sigma_3(\frac{n}{4})\\
&\quad + \frac{33}{7936}c(n) + \frac{99}{496}d(n) + \frac{63}{31}e(n),\\
U_{5,3}(n) &= \frac{1}{1999872}\sigma_9(n) + \frac{1}{31744}\sigma_9(\frac{n}{2}) + \frac{1}{465}\sigma_9(\frac{n}{4}) - \frac{1}{240}\sigma_5(\frac{n}{4}) + \frac{1}{504}\sigma_3(n)\\
&\quad - \frac{63}{31744}c(n) - \frac{99}{1984}d(n) - \frac{33}{124}e(n),\\
U_{7,1}(n) &= \frac{1}{3809280}\sigma_9(n) + \frac{17}{253952}\sigma_9(\frac{n}{2}) + \frac{17}{744}\sigma_9(\frac{n}{4}) + \frac{(2-3n)}{48}\sigma_7(\frac{n}{4})\\
&\quad - \frac{1}{480}\sigma_1(n) + \frac{529}{253952}c(n) + \frac{625}{15872}d(n) + \frac{109}{992}e(n),
\end{aligned}
$$

$$
\begin{aligned}
U_{1,9}(n) &= -\frac{7}{16584}\sigma_{11}(n) + \frac{23}{5528}\sigma_{11}(\frac{n}{2}) + \frac{256}{22803}\sigma_{11}(\frac{n}{4}) + \frac{(10-3n)}{240}\sigma_9(n) \\
&\quad + \frac{1}{264}\sigma_1(\frac{n}{4}) - \frac{1589}{55280}\tau(n) - \frac{5790}{691}\tau(\frac{n}{2}) + \frac{2562304}{691}\tau(\frac{n}{4}) + 61440f(n), \\
U_{3,7}(n) &= \frac{121}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{8}{10365}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(n) \\
&\quad - \frac{1}{480}\sigma_3(\frac{n}{4}) + \frac{729}{176896}\tau(n) + \frac{6003}{11056}\tau(\frac{n}{2}) - \frac{71496}{691}\tau(\frac{n}{4}) - 1920f(n), \\
U_{5,5}(n) &= \frac{1}{11144448}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{16}{43533}\sigma_{11}(\frac{n}{4}) + \frac{1}{504}\sigma_5(n) \\
&\quad + \frac{1}{504}\sigma_5(\frac{n}{4}) - \frac{351}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{5616}{691}\tau(\frac{n}{4}), \\
U_{9,1}(n) &= \frac{31}{5837568}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{31}{2073}\sigma_{11}(\frac{n}{4}) + \frac{(5-6n)}{120}\sigma_9(\frac{n}{4}) \\
&\quad + \frac{1}{264}\sigma_1(n) - \frac{671}{176896}\tau(n) - \frac{2505}{22112}\tau(\frac{n}{2}) - \frac{105314}{3455}\tau(\frac{n}{4}) - 240f(n).
\end{aligned}
$$

In this paper, we try to find some various convolution sum formulae of $U_{m,e,f}(n)$ and so obtain the following theorem :

Theorem 1.2. *Let* $n \in \mathbb{N}$. *Then we have*

(a)

$$
\begin{aligned}
U_{m,1,1}(n) &= \sum_{m<\frac{n}{4}} m\sigma_1(m)\sigma_1(n-4m) \\
&= \frac{1}{384}\left[n\left\{\sigma_3(n) + 3\sigma_3(\frac{n}{2}) + 16\sigma_3(\frac{n}{4}) - 2n\sigma_1(n) - 4(4n-1)\sigma_1(\frac{n}{4})\right\} + a(n)\right],
\end{aligned}
$$

(b)

$$
\begin{aligned}
U_{m,1,3}(n) &= \sum_{m<\frac{n}{4}} m\sigma_1(m)\sigma_3(n-4m) \\
&= \frac{1}{11520}\Big\{5n\sigma_5(n) + 15n\sigma_5(\frac{n}{2}) + 64n\sigma_5(\frac{n}{4}) - 18n^2\sigma_3(n) - 12n\sigma_1(\frac{n}{4}) \\
&\qquad +28b(n) + 512b(\frac{n}{2}) - 15na(n)\Big\},
\end{aligned}
$$

(c)

$$
\begin{aligned}
U_{m,3,1}(n) &= \sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_1(n-4m) \\
&= \frac{1}{23040}\Big\{n\sigma_5(n) + 15n\sigma_5(\frac{n}{2}) + 320n\sigma_5(\frac{n}{4}) - 48n\,(12n-5)\,\sigma_3(\frac{n}{4}) - 16b(n) \\
&\qquad -224b(\frac{n}{2}) + 15na(n)\Big\},
\end{aligned}
$$

(d)

$$
\begin{aligned}
U_{m,3,3}(n) &= \sum_{m<\frac{n}{4}} m\sigma_3(m)\sigma_3(n-4m) \\
&= \frac{1}{261120}\Big\{n\sigma_7(n) + 15n\sigma_7(\frac{n}{2}) + 256n\sigma_7(\frac{n}{4}) - 272n\sigma_3(\frac{n}{4}) - 272d(n) \\
&\qquad -139264d(\frac{n}{2}) - 136c(n) - 4352c(\frac{n}{2}) + 135nb(n) + 2160nb(\frac{n}{2})\Big\},
\end{aligned}
$$

(e)

$$U_{m,5,1}(n) = \sum_{m<\frac{n}{4}} m\sigma_5(m)\sigma_1(n-4m)$$
$$= \frac{1}{731136}\Big\{n\sigma_7(n) + 63n\sigma_7(\frac{n}{2}) + 5376n\sigma_7(\frac{n}{4}) - 1088n\,(12n-7)\,\sigma_5(\frac{n}{4})$$
$$+2992d(n) + 95744d(\frac{n}{2}) + 272c(n) + 2992c(\frac{n}{2}) - 273nb(n) - 3192nb(\frac{n}{2})\Big\},$$

(f)

$$U_{m,1,7}(n) = \sum_{m<\frac{n}{4}} m\sigma_1(m)\sigma_7(n-4m)$$
$$= \frac{1}{2467699200}\Big\{89280\sigma_{11}(n) - 89280\sigma_{11}(\frac{n}{2}) + 176205n\sigma_9(n) + 528615n\sigma_9(\frac{n}{2})$$
$$+ 2122752n\sigma_9(\frac{n}{4}) - 1071050n^2\sigma_7(n) - 1285260n\sigma_1(\frac{n}{4}) + 4194920\tau(n)$$
$$+ 802091520\tau(\frac{n}{2}) - 235846238208\tau(\frac{n}{4}) - 4043078369280f(n)$$
$$+215426160nd(n) - 16844037120nd(\frac{n}{2}) - 3389355nc(n) - 526376160nc(\frac{n}{2})\Big\},$$

(g)

$$U_{m,7,1}(n) = \sum_{m<\frac{n}{4}} m\sigma_7(m)\sigma_1(n-4m)$$
$$= \frac{1}{39483187200}\Big\{22320\sigma_{11}(n) - 22320\sigma_{11}(\frac{n}{2}) + 2073n\sigma_9(n) + 528615n\sigma_9(\frac{n}{2})$$
$$+ 180433920n\sigma_9(\frac{n}{4}) - 137094400n\,(4n-3)\,\sigma_7(\frac{n}{4}) - 16473648\tau(n)$$
$$- 539786880\tau(\frac{n}{2}) - 130733219840\tau(\frac{n}{4}) - 1010769592320f(n)$$
$$+202490640nd(n) + 3470699520nd(\frac{n}{2}) + 16449255nc(n) + 108459360nc(\frac{n}{2})\Big\}.$$

In addition, using Theorem 1.2 we can easily obtain some identities induced from $U_{m,e,f}(n)$, which are introduced in the following theorem :

Theorem 1.3. *For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$L(q)L(q^4)N(q)$$
$$= 1 - \frac{510}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{1530}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} - \frac{6144}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{4n}$$
$$+ \frac{3816}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^n + \frac{4536}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} + \frac{36864}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{4n}$$
$$- 432\sum_{n=1}^{\infty}n^2\sigma_5(n)q^n - \frac{444960}{31}\sum_{n=1}^{\infty}d(n)q^n + \frac{30468096}{31}\sum_{n=1}^{\infty}d(n)q^{2n}$$
$$+ \frac{3672}{31}\sum_{n=1}^{\infty}c(n)q^n + \frac{952128}{31}\sum_{n=1}^{\infty}c(n)q^{2n} - \frac{7182}{17}\sum_{n=1}^{\infty}nb(n)q^n$$
$$- \frac{314496}{17}\sum_{n=1}^{\infty}nb(n)q^{2n},$$

(b)

$$
\begin{aligned}
L^2(q^4)N(q) = 1 &- \frac{63}{62}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{945}{62}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} - \frac{7680}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{4n} \\
&+ \frac{378}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^n + \frac{2268}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} + \frac{18432}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{4n} \\
&- 108\sum_{n=1}^{\infty}n^2\sigma_5(n)q^n - \frac{63720}{31}\sum_{n=1}^{\infty}d(n)q^n - \frac{3055104}{31}\sum_{n=1}^{\infty}d(n)q^{2n} \\
&- \frac{12771}{62}\sum_{n=1}^{\infty}c(n)q^n - \frac{95472}{31}\sum_{n=1}^{\infty}c(n)q^{2n} - \frac{3591}{17}\sum_{n=1}^{\infty}nb(n)q^n \\
&- \frac{157248}{17}\sum_{n=1}^{\infty}nb(n)q^{2n},
\end{aligned}
$$

(c)

$$
\begin{aligned}
L(q)L^2(q^4) = 1 &- \frac{3}{2}\sum_{n=1}^{\infty}\sigma_5(n)q^n - \frac{45}{2}\sum_{n=1}^{\infty}\sigma_5(n)q^{2n} - 480\sum_{n=1}^{\infty}\sigma_5(n)q^{4n} \\
&+ 18\sum_{n=1}^{\infty}n\sigma_3(n)q^n + 108\sum_{n=1}^{\infty}n\sigma_3(n)q^{2n} + 4032\sum_{n=1}^{\infty}n\sigma_3(n)q^{4n} \\
&- 36\sum_{n=1}^{\infty}n^2\sigma_1(n)q^n - 4608\sum_{n=1}^{\infty}n^2\sigma_1(n)q^{4n} - \frac{9}{2}\sum_{n=1}^{\infty}a(n)q^n,
\end{aligned}
$$

(d)

$$
\begin{aligned}
L(q)L(q^4)M(q) = 1 &+ \frac{504}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{1512}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{6144}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n} \\
&- 312\sum_{n=1}^{\infty}n\sigma_5(n)q^n - 360\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} - 3072\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} \\
&+ 432\sum_{n=1}^{\infty}n^2\sigma_3(n)q^n - \frac{1932}{17}\sum_{n=1}^{\infty}b(n)q^n - \frac{52608}{17}\sum_{n=1}^{\infty}b(n)q^{2n} \\
&+ 180\sum_{n=1}^{\infty}na(n)q^n,
\end{aligned}
$$

(e)

$$
\begin{aligned}
L^2(q^4)M(q) = 1 &+ \frac{30}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{450}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{7680}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n} \\
&- 30\sum_{n=1}^{\infty}n\sigma_5(n)q^n - 180\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} - 1536\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} \\
&+ 108\sum_{n=1}^{\infty}n^2\sigma_3(n)q^n + \frac{1194}{17}\sum_{n=1}^{\infty}b(n)q^n + \frac{12576}{17}\sum_{n=1}^{\infty}b(n)q^{2n} \\
&+ 90\sum_{n=1}^{\infty}na(n)q^n,
\end{aligned}
$$

(f)

$$
\begin{aligned}
&L(q)L(q^4)M(q^4)\\
&= 1 + \frac{3}{34}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{189}{34}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{8064}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n} - \frac{3}{4}\sum_{n=1}^{\infty}n\sigma_5(n)q^n\\
&\quad - \frac{45}{2}\sum_{n=1}^{\infty}n\sigma_5(n)q^{2n} - 4992\sum_{n=1}^{\infty}n\sigma_5(n)q^{4n} + 6912\sum_{n=1}^{\infty}n^2\sigma_3(n)q^{4n}\\
&\quad - \frac{411}{34}\sum_{n=1}^{\infty}b(n)q^n - \frac{1932}{17}\sum_{n=1}^{\infty}b(n)q^{2n} - \frac{45}{4}\sum_{n=1}^{\infty}na(n)q^n,
\end{aligned}
$$

(g)

$$
\begin{aligned}
&L(q^2)M(q)M(q^4)\\
&= 1 - \frac{15}{62}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{993}{62}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} - \frac{7680}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{4n} + \frac{45}{34}\sum_{n=1}^{\infty}n\sigma_7(n)q^n\\
&\quad + \frac{675}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} + \frac{23040}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{4n} + \frac{59760}{31}\sum_{n=1}^{\infty}d(n)q^n\\
&\quad - \frac{1912320}{31}\sum_{n=1}^{\infty}d(n)q^{2n} + \frac{3735}{62}\sum_{n=1}^{\infty}c(n)q^n - \frac{59760}{31}\sum_{n=1}^{\infty}c(n)q^{2n}\\
&\quad + \frac{6075}{34}\sum_{n=1}^{\infty}nb(n)q^n + \frac{97200}{17}\sum_{n=1}^{\infty}nb(n)q^{2n},
\end{aligned}
$$

(h)

$$
\begin{aligned}
&L(q)L(q^4)N(q^4)\\
&= 1 - \frac{3}{992}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{765}{992}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} - \frac{8160}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{4n}\\
&\quad + \frac{9}{272}\sum_{n=1}^{\infty}n\sigma_7(n)q^n + \frac{567}{136}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} + \frac{61056}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{4n}\\
&\quad - 6912\sum_{n=1}^{\infty}n^2\sigma_5(n)q^{4n} - \frac{12177}{62}\sum_{n=1}^{\infty}d(n)q^n - \frac{58752}{31}\sum_{n=1}^{\infty}d(n)q^{2n}\\
&\quad - \frac{14877}{992}\sum_{n=1}^{\infty}c(n)q^n - \frac{1836}{31}\sum_{n=1}^{\infty}c(n)q^{2n} - \frac{2457}{272}\sum_{n=1}^{\infty}nb(n)q^n\\
&\quad - \frac{3591}{17}\sum_{n=1}^{\infty}nb(n)q^{2n},
\end{aligned}
$$

(i)

$$\begin{aligned} &L(q)L(q^4)M^2(q) \\ &= 1 - \frac{120}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{16488}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{49152}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n} \\ &\quad - \frac{15336}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{3672}{31}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{147456}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n} \\ &\quad + 240\sum_{n=1}^{\infty}n^2\sigma_7(n)q^n - \frac{223284}{3455}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{21161088}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} \\ &\quad + \frac{58349961216}{3455}\sum_{n=1}^{\infty}\tau(n)q^{4n} + 276037632\sum_{n=1}^{\infty}f(n)q^n - \frac{748224}{31}\sum_{n=1}^{\infty}nd(n)q^n \\ &\quad + \frac{117006336}{31}\sum_{n=1}^{\infty}nd(n)q^{2n} + \frac{11772}{31}\sum_{n=1}^{\infty}nc(n)q^n + \frac{3656448}{31}\sum_{n=1}^{\infty}nc(n)q^{2n}, \end{aligned}$$

(j)

$$\begin{aligned} &L(q)L(q^4)M^2(q^4) \\ &= 1 + \frac{33}{1382}\sum_{n=1}^{\infty}\sigma_{11}(n)q^n + \frac{63}{1382}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{2n} + \frac{65472}{691}\sum_{n=1}^{\infty}\sigma_{11}(n)q^{4n} \\ &\quad - \frac{9}{9920}\sum_{n=1}^{\infty}n\sigma_9(n)q^n - \frac{459}{992}\sum_{n=1}^{\infty}n\sigma_9(n)q^{2n} - \frac{245376}{155}\sum_{n=1}^{\infty}n\sigma_9(n)q^{4n} \\ &\quad + 3840\sum_{n=1}^{\infty}n^2\sigma_7(n)q^{4n} - \frac{116253}{6910}\sum_{n=1}^{\infty}\tau(n)q^n - \frac{332748}{691}\sum_{n=1}^{\infty}\tau(n)q^{2n} \\ &\quad - \frac{469588224}{3455}\sum_{n=1}^{\infty}\tau(n)q^{4n} - 1078272\sum_{n=1}^{\infty}f(n)q^n - \frac{10989}{124}\sum_{n=1}^{\infty}nd(n)q^n \\ &\quad - \frac{94176}{31}\sum_{n=1}^{\infty}nd(n)q^{2n} - \frac{14283}{1984}\sum_{n=1}^{\infty}nc(n)q^n - \frac{2943}{31}\sum_{n=1}^{\infty}nc(n)q^{2n}, \end{aligned}$$

(k)

$$L(q^4)A(q) = \frac{1}{2}\sum_{n=1}^{\infty}b(n)q^n + 4\sum_{n=1}^{\infty}b(n)q^{2n} + \frac{1}{2}\sum_{n=1}^{\infty}na(n)q^n.$$

2 Proof of Theorem 1.1

We introduce Proposition 2.1 and Proposition 2.2 to obtain some convolution sum formulae.

Proposition 2.1. *(See [7] For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$L(q)M(q^4) = 4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q),$$

(b)

$$N(q)L(q^2) = \frac{1}{2}L(q)N(q) - \frac{43}{170}M^2(q) + \frac{64}{85}M^2(q^2) - \frac{2016}{17}B(q),$$

(c)

$$L(q)N(q^2) = 2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q),$$

(d)

$$L(q^4)N(q) = \frac{1}{4}L(q)N(q) - \frac{16}{85}M^2(q) + \frac{63}{340}M^2(q^2) + \frac{64}{85}M^2(q^4) - \frac{4788}{17}B(q) - \frac{104832}{17}B(q^2),$$

(e)

$$M(q)M^2(q^4) = \frac{57}{416}M^3(q) + \frac{515}{104}M^3(q^2) + \frac{13932}{13}M^3(q^4) - \frac{175}{1248}N^2(q) - \frac{450}{91}N^2(q^2) - \frac{292300}{273}N^2(q^4) + 13824000F(q).$$

Proposition 2.2. *(See ([8], Theorem 1.1)) For $q \in \mathbb{C}$ with $|q| < 1$, we have*

(a)

$$L(q)L(q^2) = 1 - 72\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 48\sum_{n=1}^{\infty}\sigma_3(n)q^n - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^{2n} + 192\sum_{n=1}^{\infty}\sigma_3(n)q^{2n},$$

(b)

$$L(q)L(q^4) = 1 - 36\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 12\sum_{n=1}^{\infty}\sigma_3(n)q^n + 36\sum_{n=1}^{\infty}\sigma_3(n)q^{2n} - 576\sum_{n=1}^{\infty} n\sigma_1(n)q^{4n} + 192\sum_{n=1}^{\infty}\sigma_3(n)q^{4n},$$

(c)

$$L(q)M(q^2) = 1 - 24\sum_{n=1}^{\infty}\sigma_5(n)q^n + 1440\sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 480\sum_{n=1}^{\infty}\sigma_5(n)q^{2n},$$

(d)

$$L(q^2)M(q) = 1 + 360\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n},$$

(e)

$$L(q)L(q^2)L(q^4) = 1 - 72\sum_{n=1}^{\infty} n^2\sigma_1(n)q^n + 54\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 6\sum_{n=1}^{\infty}\sigma_5(n)q^n - 576\sum_{n=1}^{\infty} n^2\sigma_1(n)q^{2n} + 684\sum_{n=1}^{\infty} n\sigma_3(n)q^{2n} - 114\sum_{n=1}^{\infty}\sigma_5(n)q^{2n} - 4608\sum_{n=1}^{\infty} n^2\sigma_1(n)q^{4n} + 3456\sum_{n=1}^{\infty} n\sigma_3(n)q^{4n} - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{4n}.$$

Lemma 2.1. *For $q \in \mathbb{C}$ with $|q| < 1$, we obtain*

$$
\begin{aligned}
&L(q)L(q^4)N(q^2) \\
&= 1 - \frac{6}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^n - \frac{2034}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{2n} - \frac{6144}{31}\sum_{n=1}^{\infty}\sigma_9(n)q^{4n} + \frac{36}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^n \\
&\quad + \frac{12096}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{2n} + \frac{36864}{17}\sum_{n=1}^{\infty}n\sigma_7(n)q^{4n} - 1728\sum_{n=1}^{\infty}n^2\sigma_5(n)q^{2n} \\
&\quad - \frac{14688}{31}\sum_{n=1}^{\infty}d(n)q^n + \frac{470016}{31}\sum_{n=1}^{\infty}d(n)q^{2n} - \frac{459}{31}\sum_{n=1}^{\infty}c(n)q^n \\
&\quad + \frac{14688}{31}\sum_{n=1}^{\infty}c(n)q^{2n} - \frac{189}{17}\sum_{n=1}^{\infty}nb(n)q^n - \frac{6048}{17}\sum_{n=1}^{\infty}nb(n)q^{2n}.
\end{aligned}
$$

Proof. From (1.4) and Proposition 2.2 (b), we observe that

$$
\begin{aligned}
&L(q)L(q^4)N(q^2) = L(q)L(q^4)\cdot N(q^2) \\
&= \Bigg(1 - 36\sum_{n=1}^{\infty}n\sigma_1(n)q^n + 12\sum_{n=1}^{\infty}\sigma_3(n)q^n + 36\sum_{n=1}^{\infty}\sigma_3(n)q^{2n} - 576\sum_{n=1}^{\infty}n\sigma_1(n)q^{4n} \\
&\quad +192\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}\Bigg)\Bigg(1 - 504\sum_{m=1}^{\infty}\sigma_5(m)q^{2m}\Bigg) \\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4}) \\
&\quad - 504\sigma_5(\frac{N}{2}) + 36\cdot 504\sum_{m<\frac{N}{2}}(N-2m)\,\sigma_1(N-2m)\sigma_5(m) \\
&\quad - 12\cdot 504\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_5(m) - 36\cdot 504\sum_{m<\frac{N}{2}}\sigma_3(\frac{N}{2}-m)\sigma_5(m) \\
&\quad +576\cdot 504\sum_{n<\frac{N}{4}}n\sigma_1(n)\sigma_5(\frac{N}{2}-2n) - 192\cdot 504\sum_{n<\frac{N}{4}}\sigma_3(n)\sigma_5(\frac{N}{2}-2n)\Bigg\}q^N \\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4}) \\
&\quad - 504\sigma_5(\frac{N}{2}) + 36\cdot 504N\cdot T_{5,1}(N) - 36\cdot 504\cdot 2\cdot T_{m,5,1}(N) - 12\cdot 504\cdot T_{5,3}(N) \\
&\quad -36\cdot 504\cdot I_{3,5}(\frac{N}{2}) + 576\cdot 504\cdot T_{m,1,5}(\frac{N}{2}) - 192\cdot 504\cdot T_{3,5}(\frac{N}{2})\Bigg\}q^N.
\end{aligned}
$$

So we refer to Proposition 1.2 (b).

□

In ([9], Lemma 3.1 (b)) we can find

$$
\sum_{m<\frac{n}{2}}\sigma_1(m)b(n-2m) = -\frac{1}{96}\left\{32d(n) + c(n) + (3n-4)\,b(n)\right\}. \tag{2.1}
$$

Lemma 2.2. *Let* $n \in \mathbb{N}$*. Then we have*

(a)

$$\sum_{m<\frac{n}{4}} \sigma_1(m)b(n-4m) = -\frac{1}{41664}\Big\{270848e(n) + 47744d(n) - 1083392d(\frac{n}{2}) \\ +1085c(n) - 33856c(\frac{n}{2}) + 217\,(3n-8)\,b(n)\Big\},$$

(b)

$$U_{m,1,5}(n) = \sum_{m<\frac{n}{4}} m\sigma_1(m)\sigma_5(n-4m) \\ = \frac{1}{274176}\Big\{42n\sigma_7(n) + 126n\sigma_7(\frac{n}{2}) + 512n\sigma_7(\frac{n}{4}) - 204n^2\sigma_5(n) + 136n\sigma_1(\frac{n}{4}) \\ +417792e(n) + 35904d(n) + 561c(n) - 399nb(n) - 8736nb(\frac{n}{2})\Big\}.$$

Proof. (a) From Proposition 2.1 (c) we obtain

$$L(q)L(q^4)N(q^2) = L(q)N(q^2)\cdot L(q^4) \\ = \left(2L(q^2)N(q^2) + \frac{1}{85}M^2(q) - \frac{86}{85}M^2(q^2) - \frac{504}{17}B(q)\right)L(q^4) \\ = \Bigg\{2\left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + 480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}\right) + \frac{1}{85}\left(1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right) \\ -\frac{86}{85}\left(1 + 480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}\right) - \frac{504}{17}\sum_{n=1}^{\infty} b(n)q^n\Bigg\}\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right),$$

where we use (1.2) and (1.6). This shows that

$$
\begin{aligned}
&L(q)L(q^4)N(q^2)\\
&= \left(1 - 2016\sum_{n=1}^{\infty} n\sigma_5(n)q^{2n} + \frac{8064}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n} + \frac{96}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n\right.\\
&\quad \left. -\frac{504}{17}\sum_{n=1}^{\infty} b(n)q^n\right)\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1008N\sigma_5(\frac{N}{2}) + \frac{8064}{17}\sigma_7(\frac{N}{2}) + \frac{96}{17}\sigma_7(N) - \frac{504}{17}b(N) - 24\sigma_1(\frac{N}{4})\right.\\
&\quad + 2016\cdot 24\sum_{m<\frac{N}{4}}\left(\frac{N}{2} - 2m\right)\sigma_5(\frac{N}{2} - 2m)\sigma_1(m)\\
&\quad - \frac{8064}{17}\cdot 24\sum_{m<\frac{N}{2}}\sigma_7(\frac{N}{2} - 2m)\sigma_1(m) - \frac{96}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_7(N - 4m)\sigma_1(m) \qquad (2.2)\\
&\quad \left. +\frac{504}{17}\cdot 24\sum_{m<\frac{N}{4}} b(N - 4m)\sigma_1(m)\right\} q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1008N\sigma_5(\frac{N}{2}) + \frac{8064}{17}\sigma_7(\frac{N}{2}) + \frac{96}{17}\sigma_7(N) - \frac{504}{17}b(N) - 24\sigma_1(\frac{N}{4})\right.\\
&\quad + 2016\cdot 24\cdot\frac{N}{2}\cdot T_{1,5}(\frac{N}{2}) - 2016\cdot 24\cdot 2\cdot T_{m,1,5}(\frac{N}{2}) - \frac{8064}{17}\cdot 24\cdot T_{1,7}(\frac{N}{2})\\
&\quad \left. -\frac{96}{17}\cdot 24\cdot U_{1,7}(N) + \frac{504}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_1(m)b(N - 4m)\right\} q^N.
\end{aligned}
$$

Thus we equate (2.2) with Lemma 2.1 and we refer to Proposition 1.2 (b).

(b) By (1.2) and (1.12) we have

$$
\begin{aligned}
&L(q)L(q^4)N(q) = L(q)N(q)\cdot L(q^4)\\
&= \left(1 - 1008\sum_{n=1}^{\infty} n\sigma_5(n)q^n + 480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1008N\sigma_5(N) + 480\sigma_7(N) - 24\sigma_1(\frac{N}{4})\right.\\
&\qquad + 1008\cdot 24\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_5(N-4m)\sigma_1(m)\\
&\qquad \left.-480\cdot 24\sum_{m<\frac{N}{4}}\sigma_7(N-4m)\sigma_1(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-1008N\sigma_5(N) + 480\sigma_7(N) - 24\sigma_1(\frac{N}{4}) + 1008\cdot 24N\cdot U_{1,5}(N)\right.\\
&\qquad \left.-1008\cdot 24\cdot 4\sum_{m<\frac{N}{4}} m\sigma_1(m)\sigma_5(N-4m) - 480\cdot 24\cdot U_{1,7}(N)\right\}q^N.
\end{aligned} \tag{2.3}
$$

On the other hand, by (1.4) and Proposition 2.2 (b), we obtain

$$
\begin{aligned}
&L(q)L(q^4)N(q) = N(q)\cdot L(q)L(q^4)\\
&= \left(1 - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1 - 36\sum_{m=1}^{\infty} m\sigma_1(m)q^m + 12\sum_{m=1}^{\infty}\sigma_3(m)q^m\right.\\
&\qquad \left.+36\sum_{m=1}^{\infty}\sigma_3(m)q^{2m} - 576\sum_{m=1}^{\infty} m\sigma_1(m)q^{4m} + 192\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4})\right.\\
&\qquad - 504\sigma_5(N) + 504\cdot 36\sum_{m=1}^{N-1}\sigma_5(N-m)\cdot m\sigma_1(m)\\
&\qquad - 504\cdot 12\sum_{m=1}^{N-1}\sigma_5(N-m)\sigma_3(m) - 504\cdot 36\sum_{m<\frac{N}{2}}\sigma_5(N-2m)\sigma_3(m)\\
&\qquad \left.+504\cdot 576\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\cdot m\sigma_1(m) - 504\cdot 192\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\sigma_3(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4})\right.\\
&\qquad - 504\sigma_5(N) + 504\cdot 36\cdot I_{m,1,5}(N) - 504\cdot 12\cdot I_{3,5}(N) - 504\cdot 36\cdot T_{3,5}(N)\\
&\qquad \left.+504\cdot 576\sum_{m<\frac{N}{4}} m\sigma_1(m)\sigma_5(N-4m) - 504\cdot 192\cdot U_{3,5}(N)\right\}q^N.
\end{aligned} \tag{2.4}
$$

So we equate (2.3) with (2.4) and refer to Proposition 1.1 (c).

□

Proof of Theorem 1.1. By (1.4) and (1.5) we note that

$$
\begin{aligned}
L^2(q^4)N(q) &= N(q)\cdot L^2(q^4)\\
&= \left(1-504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1-288\sum_{m=1}^{\infty}m\sigma_1(m)q^{4m}+240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-72N\sigma_1(\frac{N}{4})+240\sigma_3(\frac{N}{4})-504\sigma_5(N)\\
&\quad +504\cdot 288\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\cdot m\sigma_1(m)\\
&\quad -504\cdot 240\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\sigma_3(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-72N\sigma_1(\frac{N}{4})+240\sigma_3(\frac{N}{4})-504\sigma_5(N)\\
&\quad +504\cdot 288\sum_{m<\frac{N}{4}}m\sigma_1(m)\sigma_5(N-4m)-504\cdot 240\cdot U_{3,5}(N)\Bigg\}q^N.
\end{aligned}
\tag{2.5}
$$

And by Proposition 2.1 (d) we have

$$
\begin{aligned}
L^2(q^4)N(q) &= L(q^4)N(q)\cdot L(q^4)\\
&= \Bigg(\frac{1}{4}L(q)N(q)-\frac{16}{85}M^2(q)+\frac{63}{340}M^2(q^2)+\frac{64}{85}M^2(q^4)-\frac{4788}{17}B(q)\\
&\quad -\frac{104832}{17}B(q^2)\Bigg)L(q^4)\\
&= \Bigg\{\frac{1}{4}\left(1-1008\sum_{n=1}^{\infty}n\sigma_5(n)q^n+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)-\frac{16}{85}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\\
&\quad +\frac{63}{340}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}\right)+\frac{64}{85}\left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^{4n}\right)\\
&\quad -\frac{4788}{17}\sum_{n=1}^{\infty}b(n)q^n-\frac{104832}{17}\sum_{n=1}^{\infty}b(n)q^{2n}\Bigg\}\left(1-24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right),
\end{aligned}
$$

where we use (1.6) and (1.12). This concludes that

$$
\begin{aligned}
&L^2(q^4)N(q) \\
&= \left(1 - 252\sum_{n=1}^{\infty} n\sigma_5(n)q^n + \frac{504}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^n + \frac{1512}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{2n}\right. \\
&\quad \left. + \frac{6144}{17}\sum_{n=1}^{\infty}\sigma_7(n)q^{4n} - \frac{4788}{17}\sum_{n=1}^{\infty} b(n)q^n - \frac{104832}{17}\sum_{n=1}^{\infty} b(n)q^{2n}\right) \\
&\quad \times \left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right) \\
&= 1 + \sum_{N=1}^{\infty}\left\{-252N\sigma_5(N) + \frac{504}{17}\sigma_7(N) + \frac{1512}{17}\sigma_7(\frac{N}{2}) + \frac{6144}{17}\sigma_7(\frac{N}{4}) - \frac{4788}{17}b(N)\right. \\
&\quad - \frac{104832}{17}b(\frac{N}{2}) - 24\sigma_1(\frac{N}{4}) + 252\cdot 24\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_5(N-4m)\sigma_1(m) \\
&\quad - \frac{504}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_7(N-4m)\sigma_1(m) - \frac{1512}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_7(\frac{N}{2}-2m)\sigma_1(m) \\
&\quad - \frac{6144}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_7(\frac{N}{4}-m)\sigma_1(m) + \frac{4788}{17}\cdot 24\sum_{m<\frac{N}{4}} b(N-4m)\sigma_1(m) \\
&\quad \left. + \frac{104832}{17}\cdot 24\sum_{m<\frac{N}{4}} b(\frac{N}{2}-2m)\sigma_1(m)\right\} q^N \\
&= 1 + \sum_{N=1}^{\infty}\left\{-252N\sigma_5(N) + \frac{504}{17}\sigma_7(N) + \frac{1512}{17}\sigma_7(\frac{N}{2}) + \frac{6144}{17}\sigma_7(\frac{N}{4}) - \frac{4788}{17}b(N)\right. \\
&\quad - \frac{104832}{17}b(\frac{N}{2}) - 24\sigma_1(\frac{N}{4}) + 252\cdot 24N\cdot U_{1,5}(N) \\
&\quad - 252\cdot 24\cdot 4\sum_{m<\frac{N}{4}} m\sigma_1(m)\sigma_5(N-4m) - \frac{504}{17}\cdot 24\cdot U_{1,7}(N) \\
&\quad - \frac{1512}{17}\cdot 24\cdot T_{1,7}(\frac{N}{2}) - \frac{6144}{17}\cdot 24\cdot I_{1,7}(\frac{N}{4}) + \frac{4788}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_1(m)b(N-4m) \\
&\quad \left. + \frac{104832}{17}\cdot 24\sum_{m<\frac{N}{4}}\sigma_1(m)b(\frac{N}{2}-2m)\right\} q^N.
\end{aligned} \tag{2.6}
$$

So we equate (2.6) with (2.5) and use (2.1) and Lemma 2.2 (a), then we again obtain

$$
\begin{aligned}
U_{m,1,5}(n) &= \sum_{m<\frac{n}{4}} m\sigma_1(m)\sigma_5(n-4m) \\
&= \frac{1}{59496192}\left\{9114n\sigma_7(n) + 27342n\sigma_7(\frac{n}{2}) + 111104n\sigma_7(\frac{n}{4}) - 44268n^2\sigma_5(n)\right. \\
&\quad + 29512n\sigma_1(\frac{n}{4}) + 79458816e(n) + 6390912d(n) + 44808192d(\frac{n}{2}) \\
&\quad \left. + 121737c(n) + 1400256c(\frac{n}{2}) - 86583nb(n) - 1895712nb(\frac{n}{2})\right\},
\end{aligned} \tag{2.7}
$$

which is the same result of Lemma 2.2 (b). Therefore we equate (2.7) with Lemma 2.2 (b) and we have

$$\frac{143}{6076}\left\{8e(n)+d(n)-32d(\frac{n}{2})-c(\frac{n}{2})\right\}=0,$$

so the proof is complete.

□

Remark 2.1. By Theorem 1.1, we can rewrite $U_{1,7}(n), U_{3,5}(n), ...$, etc. as

$$\begin{aligned}
U_{1,7}(n) &= \frac{17}{11904}\sigma_9(n)+\frac{17}{3968}\sigma_9(\frac{n}{2})+\frac{8}{465}\sigma_9(\frac{n}{4})+\frac{(8-3n)}{192}\sigma_7(n)-\frac{1}{480}\sigma_1(\frac{n}{4}) \\
&+\frac{433}{248}d(n)-\frac{4232}{31}d(\frac{n}{2})-\frac{109}{3968}c(n)-\frac{529}{124}c(\frac{n}{2}), \\
U_{3,5}(n) &= \frac{1}{119040}\sigma_9(n)+\frac{1}{7936}\sigma_9(\frac{n}{2})+\frac{4}{1953}\sigma_9(\frac{n}{4})-\frac{1}{240}\sigma_5(n)+\frac{1}{504}\sigma_3(\frac{n}{4}) \\
&-\frac{27}{496}d(n)+\frac{252}{31}d(\frac{n}{2})+\frac{33}{7936}c(n)+\frac{63}{248}c(\frac{n}{2}), \\
U_{5,3}(n) &= \frac{1}{1999872}\sigma_9(n)+\frac{1}{31744}\sigma_9(\frac{n}{2})+\frac{1}{465}\sigma_9(\frac{n}{4})-\frac{1}{240}\sigma_5(\frac{n}{4})+\frac{1}{504}\sigma_3(n) \\
&-\frac{33}{1984}d(n)-\frac{33}{31}d(\frac{n}{2})-\frac{63}{31744}c(n)-\frac{33}{992}c(\frac{n}{2}), \\
U_{7,1}(n) &= \frac{1}{3809280}\sigma_9(n)+\frac{17}{253952}\sigma_9(\frac{n}{2})+\frac{17}{744}\sigma_9(\frac{n}{4})+\frac{(2-3n)}{48}\sigma_7(\frac{n}{4}) \\
&-\frac{1}{480}\sigma_1(n)+\frac{407}{15872}d(n)+\frac{109}{248}d(\frac{n}{2})+\frac{529}{253952}c(n)+\frac{109}{7936}c(\frac{n}{2}),
\end{aligned} \tag{2.8}$$

also by the help of Theorem 1.1, the convolution sums in Lemma 2.2 can be rewritten as

$$\sum_{m<\frac{n}{4}}\sigma_1(m)b(n-4m)=-\frac{1}{192}\left\{64d(n)+5c(n)+(3n-8)\,b(n)\right\} \tag{2.9}$$

and

$$\begin{aligned}
U_{m,1,5}(n) = \frac{1}{274176}&\Big\{42n\sigma_7(n)+126n\sigma_7(\frac{n}{2})+512n\sigma_7(\frac{n}{4})-204n^2\sigma_5(n) \\
&+136n\sigma_1(\frac{n}{4})-16320d(n)+1671168d(\frac{n}{2})+561c(n)+52224c(\frac{n}{2}) \\
&-399nb(n)-8736nb(\frac{n}{2})\Big\}.
\end{aligned} \tag{2.10}$$

Finally, by (2.9), we have

$$\begin{aligned}
L(q^4)B(q) &= B(q)\cdot L(q^4) \\
&= \left(\sum_{n=1}^{\infty}b(n)q^n\right)\left(1-24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right) \\
&= \sum_{N=1}^{\infty}\left\{b(N)-24\sum_{m<\frac{N}{4}}b(N-4m)\sigma_1(m)\right\}
\end{aligned}$$

and so we conclude that

$$L(q^4)B(q) = 8\sum_{n=1}^{\infty} d(n)q^n + \frac{5}{8}\sum_{n=1}^{\infty} c(n)q^n + \frac{3}{8}\sum_{n=1}^{\infty} nb(n)q^n. \tag{2.11}$$

3 The convolution sums $\sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m)$ and their induced identities

Remark 3.1. $U_{7,3}(n)$ has a typo in [3] so we calculate it again. Since

$$\begin{aligned} 240 \cdot 480 \cdot U_{7,3}(N) &= 240 \cdot 480 \sum_{m<\frac{N}{4}} \sigma_7(m)\sigma_3(N-4m) \\ &= 240 \cdot 480 \left(\sum_{n=1}^{\infty} \sigma_3(n)q^n\right)\left(\sum_{m=1}^{\infty} \sigma_7(m)q^{4m}\right) \\ &= (M(q)-1)\left(M^2(q^4)-1\right) \\ &= M(q)M^2(q^4) - M(q) - M^2(q^4) + 1, \end{aligned}$$

thus we refer to (1.3), (1.6) and Proposition 2.1 (e). Therefore, we obtain

$$\begin{aligned} U_{7,3}(n) = &-\frac{7}{2653440}\sigma_{11}(n) + \frac{1}{176896}\sigma_{11}(\frac{n}{2}) + \frac{17}{20730}\sigma_{11}(\frac{n}{4}) - \frac{1}{240}\sigma_7(\frac{n}{4}) \\ &- \frac{1}{480}\sigma_3(n) + \frac{369}{176896}\tau(n) + \frac{1641}{22112}\tau(\frac{n}{2}) + \frac{22203}{1382}\tau(\frac{n}{4}) + 120f(n). \end{aligned} \tag{3.1}$$

Proof of Theorem 1.2. (a) From (1.2) and Proposition 2.2 (a), we obtain

$$
\begin{aligned}
&L(q)L(q^2)L(q^4) = L(q)L(q^2)\cdot L(q^4)\\
&= \left(1 - 72\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 48\sum_{n=1}^{\infty}\sigma_3(n)q^n - 288\sum_{n=1}^{\infty} n\sigma_1(n)q^{2n} + 192\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right)\\
&\quad\times\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1(\frac{N}{2}) + 192\sigma_3(\frac{N}{2}) - 24\sigma_1(\frac{N}{4})\\
&\quad + 72\cdot 24\sum_{m<\frac{N}{4}}(N-4m)\sigma_1(N-4m)\sigma_1(m) - 48\cdot 24\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_1(m)\\
&\quad + 288\cdot 24\sum_{m<\frac{N}{4}}\left(\frac{N}{2}-2m\right)\sigma_1(\frac{N}{2}-2m)\sigma_1(m)\\
&\quad -192\cdot 24\sum_{m<\frac{N}{4}}\sigma_3(\frac{N}{2}-2m)\sigma_1(m)\Bigg\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{-72N\sigma_1(N) + 48\sigma_3(N) - 144N\sigma_1(\frac{N}{2}) + 192\sigma_3(\frac{N}{2}) - 24\sigma_1(\frac{N}{4})\\
&\quad + 72\cdot 24N\cdot U_{1,1}(N) - 72\cdot 24\cdot 4\sum_{m<\frac{N}{4}} m\sigma_1(N-4m)\sigma_1(m)\\
&\quad - 48\cdot 24\cdot U_{1,3}(N) + 288\cdot 24\cdot\frac{N}{2}\cdot T_{1,1}(\frac{N}{2}) - 288\cdot 24\cdot 2\cdot T_{m,1,1}(\frac{N}{2})\\
&\quad -192\cdot 24\cdot T_{1,3}(\frac{N}{2})\Bigg\}q^N.
\end{aligned}
\tag{3.2}
$$

Therefore, equating (3.2) with Proposition 2.2 (e) and also using (1.13), we obtain the proof.

(b) First by (1.2) and (1.8) we have

$$
\begin{aligned}
&L(q)L(q^4)M(q) = L(q)M(q)\cdot L(q^4)\\
&= \left(1+720\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 504\sum_{n=1}^{\infty}\sigma_5(n)q^n\right)\left(1-24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{720N\sigma_3(N) - 504\sigma_5(N) - 24\sigma_1(\frac{N}{4})\\
&\quad - 720\cdot 24\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_3(N-4m)\sigma_1(m)\\
&\quad +504\cdot 24\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\sigma_1(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{720N\sigma_3(N) - 504\sigma_5(N) - 24\sigma_1(\frac{N}{4}) - 720\cdot 24N\cdot U_{1,3}(N)\\
&\quad +720\cdot 24\cdot 4\sum_{m<\frac{N}{4}} m\sigma_1(m)\sigma_3(N-4m) + 504\cdot 24\cdot U_{1,5}(N)\Bigg\}q^N.
\end{aligned}
\tag{3.3}
$$

Second by (1.3) and Proposition 2.2 (b), we obtain

$$
\begin{aligned}
&L(q)L(q^4)M(q) = M(q)\cdot L(q)L(q^4)\\
&= \left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1-36\sum_{m=1}^{\infty}m\sigma_1(m)q^m+12\sum_{m=1}^{\infty}\sigma_3(m)q^m\right.\\
&\quad\left.+36\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}-576\sum_{m=1}^{\infty}m\sigma_1(m)q^{4m}+192\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})\right.\\
&\quad+192\sigma_3(\frac{N}{4})+240\sigma_3(N)-240\cdot 36\sum_{m=1}^{N-1}\sigma_3(N-m)\cdot m\sigma_1(m)\\
&\quad+240\cdot 12\sum_{m=1}^{N-1}\sigma_3(N-m)\sigma_3(m)+240\cdot 36\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_3(m)\\
&\quad-240\cdot 576\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\cdot m\sigma_1(m)\\
&\quad\left.+240\cdot 192\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_3(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-36N\sigma_1(N)+252\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4})\right.\\
&\quad-240\cdot 36\cdot I_{m,1,3}(N)+240\cdot 12\cdot I_{3,3}(N)+240\cdot 36\cdot T_{3,3}(N)\\
&\quad\left.-240\cdot 576\sum_{m<\frac{N}{4}}m\sigma_1(m)\sigma_3(N-4m)+240\cdot 192\cdot U_{3,3}(N)\right\}q^N.
\end{aligned}
\tag{3.4}
$$

Thus we equate (3.3) with (3.4) and use Proposition 1.1 (b).

(c) By (1.2) and (1.8), we have

$$
\begin{aligned}
&L(q)L(q^4)M(q^4) = L(q)\cdot L(q^4)M(q^4)\\
&= \left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1+720\sum_{m=1}^{\infty}m\sigma_3(m)q^{4m}-504\sum_{n=1}^{\infty}\sigma_5(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{180N\sigma_3(\frac{N}{4})-504\sigma_5(\frac{N}{4})-24\sigma_1(N)\right.\\
&\qquad\left.-24\cdot 720\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\cdot m\sigma_3(m)+24\cdot 504\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\sigma_5(m)\right\}q^N \qquad (3.5)\\
&= 1+\sum_{N=1}^{\infty}\left\{180N\sigma_3(\frac{N}{4})-504\sigma_5(\frac{N}{4})-24\sigma_1(N)\right.\\
&\qquad\left.-24\cdot 720\sum_{m<\frac{N}{4}}m\sigma_3(m)\sigma_1(N-4m)+24\cdot 504\cdot U_{5,1}(N)\right\}q^N.
\end{aligned}
$$

Also by (1.3) and Proposition 2.2 (b), we obtain

$$
\begin{aligned}
&L(q)L(q^4)M(q^4) = L(q)L(q^4)\cdot M(q^4)\\
&=\Bigg(1-36\sum_{n=1}^{\infty}n\sigma_1(n)q^n+12\sum_{n=1}^{\infty}\sigma_3(n)q^n+36\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}-576\sum_{n=1}^{\infty}n\sigma_1(n)q^{4n}\\
&\quad+192\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}\Bigg)\left(1+240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&=1+\sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4})\\
&\quad+240\sigma_3(\frac{N}{4})-36\cdot240\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_1(N-4m)\sigma_3(m)\\
&\quad+12\cdot240\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_3(m)+36\cdot240\sum_{m<\frac{N}{2}}\sigma_3(\frac{N}{2}-2m)\sigma_3(m)\\
&\quad-576\cdot240\sum_{m<\frac{N}{2}}\left(\frac{N}{4}-m\right)\sigma_1(\frac{N}{4}-m)\sigma_3(m)\\
&\quad+192\cdot240\sum_{m<\frac{N}{2}}\sigma_3(\frac{N}{4}-m)\sigma_3(m)\Bigg\}q^N\\
&=1+\sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+432\sigma_3(\frac{N}{4})\\
&\quad-36\cdot240N\cdot U_{3,1}(N)+36\cdot240\cdot4\sum_{m<\frac{N}{4}}m\sigma_3(m)\sigma_1(N-4m)\\
&\quad+12\cdot240\cdot U_{3,3}(N)+36\cdot240\cdot T_{3,3}(\frac{N}{2})-576\cdot240\cdot\frac{N}{4}\cdot I_{1,3}(\frac{N}{4})\\
&\quad+576\cdot240\cdot I_{m,3,1}(\frac{N}{4})+192\cdot240\cdot I_{3,3}(\frac{N}{4})\Bigg\}q^N.
\end{aligned}
\tag{3.6}
$$

So we equate (3.5) with (3.6) and refer to Proposition 1.1 (b).

(d) By (1.3) and Proposition 2.2 (d), we obtain

$$
\begin{aligned}
&L(q^2)M(q)M(q^4) = L(q^2)M(q)\cdot M(q^4)\\
&= \left(1+360\sum_{n=1}^{\infty} n\sigma_3(n)q^n - 120\sum_{n=1}^{\infty}\sigma_5(n)q^n - 384\sum_{n=1}^{\infty}\sigma_5(n)q^{2n}\right)\\
&\quad\times\left(1+240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5(\frac{N}{2}) + 240\sigma_3(\frac{N}{4})\\
&\quad + 360\cdot 240\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_3(N-4m)\sigma_3(m) \\
&\quad -120\cdot 240\sum_{m<\frac{N}{4}}\sigma_5(N-4m)\sigma_3(m) - 384\cdot 240\sum_{m<\frac{N}{4}}\sigma_5(\frac{N}{2}-2m)\sigma_3(m)\Bigg\}q^N \qquad (3.7)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{360N\sigma_3(N) - 120\sigma_5(N) - 384\sigma_5(\frac{N}{2}) + 240\sigma_3(\frac{N}{4})\\
&\quad + 360\cdot 240N\cdot U_{3,3}(N) - 360\cdot 240\cdot 4\sum_{m<\frac{N}{4}} m\sigma_3(m)\sigma_3(N-4m)\\
&\quad -120\cdot 240\cdot U_{3,5}(N) - 384\cdot 240\cdot T_{3,5}(\frac{N}{2})\Bigg\}q^N.
\end{aligned}
$$

And by (1.3) and Proposition 2.2 (c), we have

$$
\begin{aligned}
&L(q^2)M(q)M(q^4) = M(q)\cdot L(q^2)M(q^4)\\
&= \left(1+240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\\
&\quad\times\left(1-24\sum_{m=1}^{\infty}\sigma_5(m)q^{2m} + 1440\sum_{m=1}^{\infty}m\sigma_3(m)q^{4m} - 480\sum_{m=1}^{\infty}\sigma_5(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-24\sigma_5(\frac{N}{2}) + 360N\sigma_3(\frac{N}{4}) - 480\sigma_5(\frac{N}{4}) + 240\sigma_3(N)\\
&\quad - 240\cdot 24\sum_{m<\frac{N}{2}}\sigma_3(N-2m)\sigma_5(m) + 240\cdot 1440\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\cdot m\sigma_3(m) \qquad (3.8)\\
&\quad -240\cdot 480\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_5(m)\Bigg\}q^N\\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-24\sigma_5(\frac{N}{2}) + 360N\sigma_3(\frac{N}{4}) - 480\sigma_5(\frac{N}{4}) + 240\sigma_3(N)\\
&\quad - 240\cdot 24\cdot T_{5,3}(N) + 240\cdot 1440\sum_{m<\frac{N}{4}} m\sigma_3(m)\sigma_3(N-4m)\\
&\quad -240\cdot 480\cdot U_{5,3}(N)\Bigg\}q^N.
\end{aligned}
$$

Thus we equate (3.8) with (3.7) and refer to (2.8).

(e) By (1.2) and (1.12) we observe that

$$\begin{aligned}
&L(q)L(q^4)N(q^4) = L(q)\cdot L(q^4)N(q^4)\\
&= \left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1-1008\sum_{m=1}^{\infty}m\sigma_5(m)q^{4m}+480\sum_{m=1}^{\infty}\sigma_7(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-252N\sigma_5(\frac{N}{4})+480\sigma_7(\frac{N}{4})-24\sigma_1(N)\right.\\
&\quad\left.+24\cdot 1008\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\cdot m\sigma_5(m)-24\cdot 480\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\sigma_7(m)\right\}q^N \qquad (3.9)\\
&= 1+\sum_{N=1}^{\infty}\left\{-252N\sigma_5(\frac{N}{4})+480\sigma_7(\frac{N}{4})-24\sigma_1(N)\right.\\
&\quad\left.+24\cdot 1008\sum_{m<\frac{N}{4}}m\sigma_5(m)\sigma_1(N-4m)-24\cdot 480\cdot U_{7,1}(N)\right\}q^N.
\end{aligned}$$

Similarly, by (1.4) and Proposition 2.2 (b), we obtain

$$
\begin{aligned}
L(q)L(q^4)N(q^4) &= L(q)L(q^4)\cdot N(q^4) \\
&= \Bigg(1-36\sum_{n=1}^{\infty}n\sigma_1(n)q^n+12\sum_{n=1}^{\infty}\sigma_3(n)q^n+36\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}-576\sum_{n=1}^{\infty}n\sigma_1(n)q^{4n} \\
&\quad +192\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}\Bigg)\Bigg(1-504\sum_{m=1}^{\infty}\sigma_5(m)q^{4m}\Bigg) \\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4}) \\
&\quad -504\sigma_5(\frac{N}{4})+36\cdot 504\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_1(N-4m)\sigma_5(m) \\
&\quad -12\cdot 504\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_5(m)-36\cdot 504\sum_{m<\frac{N}{4}}\sigma_3(\frac{N}{2}-2m)\sigma_5(m) \\
&\quad +576\cdot 504\sum_{n<\frac{N}{4}}n\sigma_1(n)\sigma_5(\frac{N}{4}-n)-192\cdot 504\sum_{m<\frac{N}{4}}\sigma_3(\frac{N}{4}-m)\sigma_5(m)\Bigg\}q^N \\
&= 1+\sum_{N=1}^{\infty}\Bigg\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4}) \\
&\quad -504\sigma_5(\frac{N}{4})+36\cdot 504N\cdot U_{5,1}(N)-36\cdot 504\cdot 4\sum_{m<\frac{N}{4}}m\sigma_5(m)\sigma_1(N-4m) \\
&\quad -12\cdot 504\cdot U_{5,3}(N)-36\cdot 504\cdot T_{5,3}(\frac{N}{2})+576\cdot 504\cdot I_{m,1,5}(\frac{N}{4}) \\
&\quad -192\cdot 504\cdot I_{3,5}(\frac{N}{4})\Bigg\}q^N.
\end{aligned}
\tag{3.10}
$$

Thus we equate (3.9) with (3.10) and use Proposition 1.1 (c) and (2.8).

(f) By (1.2) and (1.10) we have

$$
\begin{aligned}
&L(q)L(q^4)M^2(q) = L(q)M^2(q)\cdot L(q^4)\\
&= \left(1 + 720\sum_{n=1}^{\infty} n\sigma_7(n)q^n - 264\sum_{n=1}^{\infty}\sigma_9(n)q^n\right)\left(1 - 24\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{720N\sigma_7(N) - 264\sigma_9(N) - 24\sigma_1(\frac{N}{4})\\
&\quad - 720\cdot 24\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_7(N-4m)\sigma_1(m)\\
&\quad +264\cdot 24\sum_{m<\frac{N}{4}}\sigma_9(N-4m)\sigma_1(m)\Bigg\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\Bigg\{720N\sigma_7(N) - 264\sigma_9(N) - 24\sigma_1(\frac{N}{4}) - 720\cdot 24N\cdot U_{1,7}(N)\\
&\quad +720\cdot 24\cdot 4\sum_{m<\frac{N}{4}} m\sigma_1(m)\sigma_7(N-4m) + 264\cdot 24\cdot U_{1,9}(N)\Bigg\}q^N.
\end{aligned}
\tag{3.11}
$$

Also by (1.6) and Proposition 2.2 (b) we obtain

$$
\begin{aligned}
&L(q)L(q^4)M^2(q) = M^2(q)\cdot L(q)L(q^4)\\
&= \left(1+480\sum_{n=1}^{\infty}\sigma_7(n)q^n\right)\left(1-36\sum_{m=1}^{\infty}m\sigma_1(m)q^m+12\sum_{m=1}^{\infty}\sigma_3(m)q^m\right.\\
&\qquad\left.+36\sum_{m=1}^{\infty}\sigma_3(m)q^{2m}-576\sum_{m=1}^{\infty}m\sigma_1(m)q^{4m}+192\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4})\right.\\
&\qquad+480\sigma_7(N)-480\cdot 36\sum_{m=1}^{N-1}\sigma_7(N-m)\cdot m\sigma_1(m)\\
&\qquad+480\cdot 12\sum_{m=1}^{N-1}\sigma_7(N-m)\sigma_3(m)+480\cdot 36\sum_{m<\frac{N}{2}}\sigma_7(N-2m)\sigma_3(m)\\
&\qquad\left.-480\cdot 576\sum_{m<\frac{N}{4}}\sigma_7(N-4m)\cdot m\sigma_1(m)+480\cdot 192\sum_{m<\frac{N}{4}}\sigma_7(N-4m)\sigma_3(m)\right\}q^N\\
&= 1+\sum_{N=1}^{\infty}\left\{-36N\sigma_1(N)+12\sigma_3(N)+36\sigma_3(\frac{N}{2})-144N\sigma_1(\frac{N}{4})+192\sigma_3(\frac{N}{4})\right.\\
&\qquad+480\sigma_7(N)-480\cdot 36\cdot I_{m,1,7}(N)+480\cdot 12\cdot I_{3,7}(N)+480\cdot 36\cdot T_{3,7}(N)\\
&\qquad\left.-480\cdot 576\sum_{m<\frac{N}{4}}m\sigma_1(m)\sigma_7(N-4m)+480\cdot 192\cdot U_{3,7}(N)\right\}q^N.
\end{aligned}
\tag{3.12}
$$

So we equate (3.11) with (3.12) and refer to Proposition 1.1 (d) and (2.8).

(g) By (1.2) and (1.10) we note

$$
\begin{aligned}
&L(q)L(q^4)M^2(q^4) = L(q)\cdot L(q^4)M^2(q^4)\\
&= \left(1-24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1+720\sum_{m=1}^{\infty}m\sigma_7(m)q^{4m}-264\sum_{m=1}^{\infty}\sigma_9(m)q^{4m}\right)\\
&= 1+\sum_{N=1}^{\infty}\left\{180N\sigma_7(\frac{N}{4})-264\sigma_9(\frac{N}{4})-24\sigma_1(N)\right.\\
&\qquad\left.-24\cdot 720\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\cdot m\sigma_7(m)+24\cdot 264\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\sigma_9(m)\right\}q^N \qquad (3.13)\\
&= 1+\sum_{N=1}^{\infty}\left\{180N\sigma_7(\frac{N}{4})-264\sigma_9(\frac{N}{4})-24\sigma_1(N)\right.\\
&\qquad\left.-24\cdot 720\sum_{m<\frac{N}{4}}m\sigma_7(m)\sigma_1(N-4m)+24\cdot 264\cdot U_{9,1}(N)\right\}q^N.
\end{aligned}
$$

Also by (1.6) and Proposition 2.2 (b) we obtain

$$
\begin{aligned}
&L(q)L(q^4)M^2(q^4) = L(q)L(q^4)\cdot M^2(q^4)\\
&= \left(1 - 36\sum_{n=1}^{\infty} n\sigma_1(n)q^n + 12\sum_{n=1}^{\infty}\sigma_3(n)q^n + 36\sum_{n=1}^{\infty}\sigma_3(n)q^{2n}\right.\\
&\qquad \left. -576\sum_{n=1}^{\infty} n\sigma_1(n)q^{4n} + 192\sum_{n=1}^{\infty}\sigma_3(n)q^{4n}\right)\left(1 + 480\sum_{m=1}^{\infty}\sigma_7(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4})\right.\\
&\qquad + 480\sigma_7(\frac{N}{4}) - 36\cdot 480\sum_{m<\frac{N}{4}}(N-4m)\,\sigma_1(N-4m)\sigma_7(m)\\
&\qquad + 12\cdot 480\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\sigma_7(m) + 36\cdot 480\sum_{m<\frac{N}{4}}\sigma_3(\frac{N}{2}-2m)\sigma_7(m)\\
&\qquad \left. -576\cdot 480\sum_{n<\frac{N}{4}} n\sigma_1(n)\sigma_7(\frac{N}{4}-n) + 192\cdot 480\sum_{m<\frac{N}{4}}\sigma_3(\frac{N}{4}-m)\sigma_7(m)\right\}q^N \\
&= 1 + \sum_{N=1}^{\infty}\left\{-36N\sigma_1(N) + 12\sigma_3(N) + 36\sigma_3(\frac{N}{2}) - 144N\sigma_1(\frac{N}{4}) + 192\sigma_3(\frac{N}{4})\right.\\
&\qquad + 480\sigma_7(\frac{N}{4}) - 36\cdot 480N\cdot U_{7,1}(N) + 36\cdot 480\cdot 4\sum_{m<\frac{N}{4}} m\sigma_7(m)\sigma_1(N-4m)\\
&\qquad + 12\cdot 480\cdot U_{7,3}(N) + 36\cdot 480\cdot T_{7,3}(\frac{N}{2}) - 576\cdot 480\cdot I_{m,1,7}(\frac{N}{4})\\
&\qquad \left. +192\cdot 480\cdot I_{3,7}(\frac{N}{4})\right\}q^N.
\end{aligned}
\tag{3.14}
$$

So we equate (3.13) with (3.14) and refer to Proposition 1.1 (d) and (3.1).

□

Now Theorem 1.2 is the based results to induce Theorem 1.3 :

Proof of Theorem 1.3. (a) Insert (2.10) into (2.3).

(b) Insert (2.10) into (2.5).

(c) By (1.2) and (1.5), we note that

$$
\begin{aligned}
&L(q)L^2(q^4)\\
&= \left(1 - 24\sum_{n=1}^{\infty}\sigma_1(n)q^n\right)\left(1 - 288\sum_{m=1}^{\infty}m\sigma_1(m)q^{4m} + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-72N\sigma_1(\frac{N}{4}) + 240\sigma_3(\frac{N}{4}) - 24\sigma_1(N)\right.\\
&\qquad\left. +24\cdot 288\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\cdot m\sigma_1(m) - 24\cdot 240\sum_{m<\frac{N}{4}}\sigma_1(N-4m)\sigma_3(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-72N\sigma_1(\frac{N}{4}) + 240\sigma_3(\frac{N}{4}) - 24\sigma_1(N) + 24\cdot 288\cdot U_{m,1,1}(N)\right.\\
&\qquad\left. -24\cdot 240\cdot U_{3,1}(N)\right\}q^N.
\end{aligned}
$$

Finally, we refer to Theorem 1.2 (a) so we complete $L(q)L^2(q^4)$.

(d) Insert Theorem 1.2 (b) into (3.3).

(e) By (1.3) and (1.5), we obtain

$$
\begin{aligned}
&L^2(q^4)M(q) = M(q)\cdot L^2(q^4)\\
&= \left(1 + 240\sum_{n=1}^{\infty}\sigma_3(n)q^n\right)\left(1 - 288\sum_{m=1}^{\infty}m\sigma_1(m)q^{4m} + 240\sum_{m=1}^{\infty}\sigma_3(m)q^{4m}\right)\\
&= 1 + \sum_{N=1}^{\infty}\left\{-72N\sigma_1(\frac{N}{4}) + 240\sigma_3(\frac{N}{4}) + 240\sigma_3(N)\right.\\
&\qquad - 240\cdot 288\sum_{m<\frac{N}{4}}\sigma_3(N-4m)\cdot m\sigma_1(m)\\
&\qquad\left. +240\cdot 240\sum_{m<\frac{N}{2}}\sigma_3(N-4m)\sigma_3(m)\right\}q^N\\
&= 1 + \sum_{N=1}^{\infty}\left\{-72N\sigma_1(\frac{N}{4}) + 240\sigma_3(\frac{N}{4}) + 240\sigma_3(N)\right.\\
&\qquad\left. -240\cdot 288\sum_{m<\frac{N}{4}}m\sigma_1(m)\sigma_3(N-4m) + 240\cdot 240\cdot U_{3,3}(N)\right\}q^N.
\end{aligned}
\tag{3.15}
$$

So we insert Theorem 1.2 (b) into (3.15).

(f) Insert Theorem 1.2 (c) into (3.5).

(g) Insert Theorem 1.2 (d) into (3.7).

(h) Insert Theorem 1.2 (e) into (3.9).

(i) Insert Theorem 1.2 (f) into (3.11).

(j) Insert Theorem 1.2 (g) into (3.13).

(k) By Proposition 2.1 (a), we can expand Theorem 1.3 (f) as

$$\begin{aligned}
&L(q)L(q^4)M(q^4) = L(q^4)\cdot L(q)M(q^4)\\
&= L(q^4)\left(4L(q^4)M(q^4) + \frac{1}{336}N(q) + \frac{5}{112}N(q^2) - \frac{64}{21}N(q^4) - \frac{45}{2}A(q)\right)\\
&= 4L^2(q^4)M(q^4) + \frac{1}{336}L(q^4)N(q) + \frac{5}{112}L(q^4)N(q^2) - \frac{64}{21}L(q^4)N(q^4)\\
&\quad - \frac{45}{2}L(q^4)A(q).
\end{aligned}$$

So we use (1.9), (1.12), Proposition 2.1 (b) and (d).

□

Corollary 3.1. *Let* $n \in \mathbb{N}$*. Then we have*

(a)

$$\sum_{m<\frac{n}{4}} \sigma_1(m)a(n-4m) = -\frac{1}{48}\left\{b(n) + 8b(\frac{n}{2}) + (n-2)\,a(n)\right\},$$

(b)

$$\begin{aligned}
&\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+4b+4c=n}} \sigma_1(a)\sigma_1(b)\sigma_1(c) = \sum_{k<\frac{n}{4}}\sum_{m=1}^{k-1}\sigma_1(n-4k)\sigma_1(k-m)\sigma_1(m)\\
&= \frac{1}{9216}\left\{\sigma_5(n) + 15\sigma_5(\frac{n}{2}) + 320\sigma_5(\frac{n}{4}) - 4\,(3n-4)\,\sigma_3(n)\right.\\
&\quad - 12\,(3n-4)\,\sigma_3(\frac{n}{2}) - 32\,(21n-13)\,\sigma_3(\frac{n}{4}) + 8\left(3n^2-6n+2\right)\sigma_1(n)\\
&\quad \left. +16\left(12n^2-15n+2\right)\sigma_1(\frac{n}{4}) + 3a(n)\right\}.
\end{aligned}$$

Proof. (a) By (1.1) and (1.2), we have

$$\begin{aligned}
24\sum_{N=1}^{\infty}\left(\sum_{m<\frac{N}{4}}\sigma_1(m)a(N-4m)\right)q^N &= 24\left(\sum_{n=1}^{\infty}a(n)q^n\right)\left(\sum_{m=1}^{\infty}\sigma_1(m)q^{4m}\right)\\
&= A(q)\left(1-L(q^4)\right)\\
&= A(q) - A(q)L(q^4).
\end{aligned}$$

So we use Theorem 1.3 (k).

(b) By (1.2), we observe that

$$\begin{aligned}
&13824\sum_{n=1}^{\infty}\left(\sum_{\substack{(a,b,c)\in\mathbb{N}^3\\ a+4b+4c=n}}\sigma_1(a)\sigma_1(b)\sigma_1(c)\right)q^n = 24^3\sum_{(a,b,c)\in\mathbb{N}^3}\sigma_1(a)\sigma_1(b)\sigma_1(c)q^{a+4b+4c}\\
&= \left(24\sum_{a=1}^{\infty}\sigma_1(a)q^a\right)\left(24\sum_{b=1}^{\infty}\sigma_1(b)q^{4b}\right)^2 = (1-L(q))\left(1-L(q^4)\right)^2\\
&= 1 - 2L(q^4) + L^2(q^4) - L(q) + 2L(q)L(q^4) - L(q)L^2(q^4).
\end{aligned}$$

So appealing to (1.2), (1.5), Theorem 1.3 (c), and Proposition 2.2 (b) we can obtain the proof. □

4 Conclusions

In this paper, we study some various convolution sum formulae mainly as the form

$$U_{m,e,f}(n) = \sum_{m<\frac{n}{4}} m\sigma_e(m)\sigma_f(n-4m)$$

for $n(\in \mathbb{N})$ and an odd positive integer e and f. In addition we can deduce some identities from these convolution sum formulae $U_{m,e,f}(n)$. Especially, we obtain the coefficient relation as

$$e(n) = -\frac{1}{8}\left\{d(n) - 32d(\frac{n}{2}) - c(\frac{n}{2})\right\}$$

in Theorem 1.1.

Competing Interests

The author declares that no competing interests exist.

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Appendix

The first twenty values of $\tau(n)$ are given in the Table 1,

n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$	n	$\tau(n)$
1	1	6	-6048	11	534612	16	987136
2	-24	7	-16744	12	-370944	17	-6905934
3	252	8	84480	13	-577738	18	2727432
4	-1472	9	-113643	14	401856	19	10661420
5	4830	10	-115920	15	1217160	20	-7109760

TABLE 1. $\tau(n)$ for n $(1 \leq n \leq 20)$

similarly the first twenty values of $a(n)$, $b(n)$, $c(n)$, $d(n)$, $e(n)$, and $f(n)$ are listed in the following tables.

n	$a(n)$	n	$a(n)$	n	$a(n)$	n	$a(n)$
1	1	6	0	11	540	16	0
2	0	7	-88	12	0	17	594
3	-12	8	0	13	-418	18	0
4	0	9	-99	14	0	19	836
5	54	10	0	15	-648	20	0

TABLE 2. $a(n)$ for n $(1 \leq n \leq 20)$

n	$b(n)$	n	$b(n)$	n	$b(n)$	n	$b(n)$
1	1	6	-96	11	1092	16	4096
2	-8	7	1016	12	768	17	14706
3	12	8	-512	13	1382	18	16344
4	64	9	-2043	14	-8128	19	-39940
5	-210	10	1680	15	-2520	20	-13440

TABLE 3. $b(n)$ for n $(1 \leq n \leq 20)$

n	$c(n)$	n	$c(n)$	n	$c(n)$	n	$c(n)$
1	1	6	2496	11	-38996	16	-65536
2	-16	7	-4536	12	39936	17	311442
3	100	8	-4096	13	37806	18	-74448
4	-256	9	23085	14	15232	19	128244
5	-154	10	-13920	15	-146472	20	-222720

TABLE 4. $c(n)$ for n $(1 \leq n \leq 20)$

n	$d(n)$	n	$d(n)$	n	$d(n)$	n	$d(n)$
1	0	6	-156	11	-536	16	4096
2	1	7	112	12	-2496	17	-17472
3	-8	8	256	13	4384	18	4653
4	16	9	-576	14	-952	19	5848
5	32	10	870	15	336	20	13920

TABLE 5. $d(n)$ for n $(1 \le n \le 20)$

n	$e(n)$	n	$e(n)$	n	$e(n)$	n	$e(n)$
1	0	6	0	11	67	16	0
2	0	7	-14	12	0	17	2184
3	1	8	0	13	-548	18	0
4	0	9	72	14	0	19	-731
5	-4	10	0	15	-42	20	0

TABLE 6. $e(n)$ for n $(1 \le n \le 20)$

n	$f(n)$	n	$f(n)$	n	$f(n)$	n	$f(n)$
1	0	6	8	11	6296	16	388608
2	0	7	44	12	16384	17	756822
3	0	8	192	13	39569	18	1419200
4	0	9	694	14	89424	19	2572328
5	1	10	2208	15	191028	20	4521984

TABLE 7. $f(n)$ for n $(1 \le n \le 20)$

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