Research Article

Reliability Estimation of Inverse Lomax Distribution Using Extreme Ranked Set Sampling

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In survival analysis, the two-parameter inverse Lomax distribution is an important lifetime distribution. In this study, the estimation of $R = P[Y < X]$ is investigated when the stress and strength random variables are independent inverse Lomax distribution. Using the maximum likelihood approach, we obtain the R estimator via simple random sample (SRS), ranked set sampling (RSS), and extreme ranked set sampling (ERSS) methods. Four different estimators are developed under the ERSS framework. Two estimators are obtained when both strength and stress populations have the same set size. The two other estimators are obtained when both strength and stress distributions have dissimilar set sizes. Through a simulation experiment, the suggested estimates are compared to the corresponding under SRS. Also, the reliability estimates via ERSS method are compared to those under RSS scheme. It is found that the reliability estimate based on RSS and ERSS schemes is more efficient than the equivalent using SRS based on the same number of measured units. The reliability estimates based on RSS scheme are more appropriate than the others in most situations. For small even set size, the reliability estimate via ERSS scheme is more efficient than those under RSS and SRS. However, in a few cases, reliability estimates via ERSS method are more accurate than using RSS and SRS schemes.

1. Introduction

The inverse Lomax (ILo) distribution is considered as the reciprocal of the Lomax distribution. In some situations, it is a good alternative to the famous distributions like gamma, inverse Weibull, and Weibull. It has varied applications in modelling several types of data, including economics and actuarial sciences (see [1]). It has an application in geophysical databases [2]. The ILo distribution has an important application in reliability analysis [3]. Statistical inference for this distribution has been discussed by several researchers (see, for example, [4, 5]). In the present work, the ILo distribution is taken under the stress strength (S-S) model associated with any system that depends on different sampling schemes. The cumulative distribution function (cdf) of the ILo distribution with shape parameter ω and scale parameter ρ is specified by the following:

$$H(x;\rho,\omega) = \left(1 + \frac{\rho}{x}\right)^{-\omega}, x, \rho, \omega > 0. \quad (1)$$

The probability density function (pdf) of the ILo distribution is as follows:

$$h(x;\rho,\omega) = \frac{\rho\omega}{x^2}\left(1 + \frac{\rho}{x}\right)^{-\omega-1}, x, \rho, \omega > 0. \quad (2)$$

The RSS was first introduced in [6] as a sampling scheme. The RSS scheme is used in situations when it is difficult and expensive to measure a large number of elements, but visually (without inspection) ranking some of them is easier and cheaper. This sampling design is both a cost-effective and powerful alternative to the commonly used SRS. This scheme involves randomly selecting m_1 sets (each of size m_1 elements) from the study population. The elements of each set are ordered with respect to the variable of the study by any negligible cost method or visually without measurements. Finally, the ath minimum from the ath set, $a = 1, 2 \cdots, m_1$, is specified for measurement. The obtained sample is called a RSS of set size m_1. The whole procedure can be repeated q times to yield a RSS of size $m^{\bullet} = qm_1$. The mathematical theory of the RSS method has been provided in [7]. Studies on RSS scheme have been proposed by several authors (see, for example, [8–15]).

Several modifications of the RSS have been proposed to improve the efficiency of the estimators. Herein, we are interested in the RSS and ERSS, presented in [16]. The ERSS procedure involves randomly selecting m_1 sets (each of size m_1 elements). The elements of each set are ordered with respect to variable of the study by visual inspection or any other cost free method. For an odd set size (OSZ), we select from the first $((m_1 - 1)/2)$ samples the smallest ranked unit, from the other $((m_1 - 1)/2)$ the largest ranked unit, and for the last sample select the median of the sample for actual measurement. For even set size (ESZ), we chose from $(m_1/2)$ samples the smallest ranked unit and from the other $(m_1/2)$ samples the largest ranked unit for actual measurement. This procedure can be repeated q times to obtain $m_1 q$ units from ERSS data.

The S-S reliability $R = P(Y < X)$ is the probability of the system working when a strength X is greater than a stress Y. So, the system will stop working when the applied stress is greater than its strength. Thus, the parameter R is a measure of a system's reliability, which has many applications in physics, engineering, genetics, psychology, and economics. There is an extensive literature on estimating R based on SRS (see, for instance, [17–24]). However, in recent years, statistical inferences about the S-S model based on the RSS method have been considered by several researchers. Reference [25] discussed estimation of S-S reliability for exponential populations. Reference [26] proposed three estimators of R when X and Y are independent exponential populations. References [11, 27] discussed the estimation of the S-S model when Y and X are two independent Burr type XII distribution under several modifications of the RSS method. Estimation of the S-S model for Weibull and Lindley distributions has been discussed, respectively, in [28, 29]. Reference [30] obtained a reliability estimator of R for the exponentiated Pareto distribution under the RSS scheme.

The S-S model is one of the important approaches in reliability analysis. The S-S model can be used to solve a variety of engineering problems, such as determining whether a building's strength should be subjected to the design earthquake, whether a rocket motor's strength should be greater than the operating pressure, and comparing the strength of different materials. The ILo is one of the distributions which is used quite effectively for modelling the strength of data used in economics, geography, actuarial, and medical fields. It has been discovered to be very flexible in analyzing situations with a realized nonmonotonic failure rate, which has wide applications in modelling life components. The RSS method and its modifications are frequently employed to gather samples that are more representative of the underlying population, when sampling units are expensive and difficult to measure but easy and inexpensive to arrange according to the variable of interest. In this method, ranking can be done using expert opinions, auxiliary variables, or any other low-cost approach. Statistical inference on the S-S model, based on the RSS scheme and its variations, has recently gotten a lot of attention. Due to the importance of the ILo distribution in reliability research, we propose to evaluate the reliability estimator of the S-S model where the strength $X \sim \text{ILo}(\rho, \omega)$ and stress $Y \sim \text{ILo}(\rho, \varphi)$ are both independent. Under SRS, RSS, and ERSS methods, the maximum likelihood (ML) estimators of R are derived. Based on the ERSS scheme, we get the ML estimator of R when both X and Y populations have similar or dissimilar set sizes. We evaluate the accuracy of estimators using absolute biases (ABs), mean squared errors (MSEs), and relative efficiencies (REs) in a simulated exercise. The remainder of this essay is structured in the following manner. In Section 2, we extract R's expression and use SRS to calculate R's ML estimator. In Section 3, the RSS is used to obtain an estimator for the S-S model. Section 4 presents reliability estimators of the S-S model using ERSS methodology. A numerical analysis is included in Section 5. Finally, in Section 6, we bring the paper to a close.

2. Estimator of R Using SRS

In this section, we derive the expression of R as well as obtain its ML estimator. Assuming that the strength X and stress Y are independently distributed random variables with the same scale parameter, where $X \sim \text{ILo}(\rho, \omega)$ and $Y \sim \text{ILo}(\rho, \varphi)$, the system's reliability with stress variable Y and strength variable X is given by the following:

$$
\begin{aligned}
R &= \int_0^{\infty} h(x) H_y(x) dx \\
&= \omega\rho \int_{x=0}^{\infty} \frac{1}{x^2}\left(1 + \frac{\rho}{x}\right)^{-\omega}\left(1 + \frac{\rho}{x}\right)^{-(\varphi+1)} dx \\
&= \frac{\omega}{\omega + \varphi}.
\end{aligned}
\tag{3}
$$

The strength-stress parameter R given in (3) depends on the shape parameters ω and φ. Let $X_1, X_2, \cdots, X_{n^{\bullet}}$ be a SRS of size $n^{\bullet}$ from the $\text{ILo}(\rho, \omega)$, and $Y_1, Y_2, \cdots, Y_{m^{\bullet}}$ of size $m^{\bullet}$ be SRS from the $\text{ILo}(\rho, \varphi)$ being independent

with a common scale parameter. The log-likelihood of the observed sample is given by

$$
\begin{aligned}
\ell = {} & n^{\bullet} \ln \omega + (n^{\bullet} + m^{\bullet}) \ln \rho + m^{\bullet} \ln \varphi \\
& - 2 \sum_{a=1}^{n^{\bullet}} \ln x_a - (\omega + 1) \left[\sum_{a=1}^{n^{\bullet}} \ln \left(1 + \frac{\rho}{x_a} \right) \right] \\
& - 2 \sum_{b=1}^{m^{\bullet}} \ln y_b - (\varphi + 1) \sum_{b=1}^{m^{\bullet}} \ln \left(1 + \frac{\rho}{y_b} \right).
\end{aligned}
\tag{4}
$$

The partial derivatives of ℓ with respect to ρ, φ, and ω are, respectively, given by

$$
\frac{\partial \ln \ell}{\partial \omega} = \frac{n^{\bullet}}{\omega} - \left[\sum_{a=1}^{n^{\bullet}} \ln \left(1 + \frac{\rho}{x_a} \right) \right],
\tag{5}
$$

$$
\frac{\partial \ln \ell}{\partial \varphi} = \frac{m^{\bullet}}{\varphi} - \left[\sum_{b=1}^{m^{\bullet}} \ln \left(1 + \frac{\rho}{y_b} \right) \right],
\tag{6}
$$

$$
\frac{\partial \ln \ell}{\partial \rho} = \frac{(n + m)}{\rho} - \left[\sum_{a=1}^{n^{\bullet}} \frac{(\omega + 1)}{(\rho + x_a)} \right] - \left[\sum_{b=1}^{m^{\bullet}} \frac{(\varphi + 1)}{(\rho + y_b)} \right].
\tag{7}
$$

Setting Equations (5)–(7) with zero and solving numerically, we get the ML estimators of ρ, φ, and ω, say $\hat{\rho}$, $\hat{\varphi}$, and $\hat{\omega}$. After that, the ML estimator of R, say $\hat{R}$, is obtained as follows:

$$
\hat{R} = \frac{\hat{\omega}}{\hat{\omega} + \hat{\varphi}}.
\tag{8}
$$

3. Estimator of R Using RSS

We derive the reliability estimator when the random samples of strength $X \sim \mathrm{ILo}(\rho, \omega)$ and stress $Y \sim \mathrm{ILo}(\rho, \varphi)$ are observed from the RSS design. Let $\{X_{a(a)e}, a = 1, 2, \cdots, m_1,\ e = 1, 2, \cdots, q_x\}$ be a RSS of size $n^{\bullet} = m_1 q_x$ for X where $X_{a(a)e}$ is the ath order statistics of size m_1 of the eth cycle.

Similarly, let $\{Y_{b(b)g}, b = 1, 2, \cdots, m_2, g = 1, 2, \cdots, q_y\}$ be a RSS method of size $m^{\bullet} = m_2 q_y$, where m_2 is the set size and q_y is the number of cycles. For simplified forms, we use the notations X_{ae} and Ybg instead of the notations $Xa(a)e$ and $Yb(b)g$, respectively, for easy understanding and the simplicity. The pdf of Xae and Ybg are given, respectively, by

$$
\begin{aligned}
h_{X_a}(x_{ae}) = {} & \frac{m_1!}{(a-1)!(m_1-a)!} (H(x_{ae}))^{a-1} \\
& \cdot [1 - H(x_{ae})]^{m_1-a} h(x_{ae}), \quad x_{ae} > 0, \\
h_{Y_b}\left(y_{bg}\right) = {} & \frac{m_2!}{(b-1)!(m_2-b)!} \left(H\left(y_{bg}\right)\right)^{b-1} \\
& \cdot \left[1 - H\left(y_{bg}\right)\right]^{m_2-b} h\left(y_{bg}\right), \quad y_{bg} > 0.
\end{aligned}
\tag{9}
$$

The likelihood function, say ℓ_1, based on RSS is given by

$$
\begin{aligned}
\ell_1 = {} & \prod_{e=1}^{q_x} \prod_{a=1}^{m_1} C_1 \frac{\omega \rho}{x_{ae}^2} \left(1 + \frac{\rho}{x_{ae}} \right)^{-(a\omega+1)} \left[1 - \left(1 + \frac{\rho}{x_{ae}} \right)^{-\omega} \right]^{m_1-a} \\
& \cdot \prod_{g=1}^{q_y} \prod_{b=1}^{m_2} C_2 \frac{\varphi \rho}{y_{bg}^2} \left(1 + \frac{\rho}{y_{bg}} \right)^{-(b\varphi+1)} \left[1 - \left(1 + \frac{\rho}{y_{bg}} \right)^{-\varphi} \right]^{m_2-b},
\end{aligned}
\tag{10}
$$

$C_1 = m_1!/(a-1)!(m_1-a)!$, $C_2 = m_2!/(b-1)!(m_2-b)!$.

The ML estimators of ω, φ, and ρ are the solutions of the following equations:

$$
\begin{aligned}
\frac{\partial \ln \ell_1}{\partial \omega} = {} & \frac{n^{\bullet}}{\omega} - \sum_{e=1}^{q_x} \sum_{a=1}^{m_1} a \ln \left(1 + \frac{\rho}{x_{ae}} \right) \\
& + \sum_{e=1}^{q_x} \sum_{a=1}^{m_1} \frac{(m_1 - a) \ln (1 + (\rho/x_{ae}))}{(1 + (\rho/x_{ae}))^{\omega} - 1} = 0,
\end{aligned}
\tag{11}
$$

$$
\begin{aligned}
\frac{\partial \ln \ell_1}{\partial \varphi} = {} & \frac{m^{\bullet}}{\varphi} - b \sum_{g=1}^{q_y} \sum_{b=1}^{m_2} \ln \left(1 + \frac{\rho}{y_{bg}} \right) \\
& + \sum_{g=1}^{q_y} \sum_{b=1}^{m_2} \frac{(m_2 - b) \ln \left(1 + \left(\rho/y_{bg} \right) \right)}{\left(1 + \left(\rho/y_{bg} \right) \right)^{\varphi} - 1} = 0,
\end{aligned}
\tag{12}
$$

$$
\begin{aligned}
\frac{\partial \ln \ell_1}{\partial \rho} = {} & \frac{m^{\bullet} + n^{\bullet}}{\rho} - \sum_{e=1}^{q_x} \sum_{a=1}^{m_1} \frac{(a\omega + 1)}{(x_{ae} + \rho)} \\
& + \sum_{e=1}^{q_x} \sum_{a=1}^{m_1} \frac{(m_1 - a)\omega(1 + (\rho/x_{ae}))^{-\omega-1}}{x_{ae}[1 - (1 + (\rho/x_{ae}))^{-\omega}]} \\
& - \sum_{g=1}^{q_y} \sum_{b=1}^{m_2} \frac{(b\varphi + 1)}{\left(y_{bg} + \rho \right)} \\
& + \sum_{g=1}^{q_y} \sum_{b=1}^{m_2} \frac{(m_2 - b)\varphi \left(1 + \left(\rho/y_{bg} \right) \right)^{-\varphi-1}}{y_{bg} \left[1 - \left(1 + \left(\rho/y_{bg} \right) \right)^{-\varphi} \right]} \\
& = 0.
\end{aligned}
\tag{13}
$$

As can be seen, we use iterative approaches to solve Equations (11)–(13) because there are no explicit solutions. As a result, the ML estimator of S-S reliability is obtained based on the invariance property of ML estimators.

4. Estimator of R Using ERSS

In this section, we obtain the ML estimator of R when strength X and stress Y have an ILo distribution under the ERSS design. In these respects, the reliability estimator is considered in two cases when both X and Y distributions have similar or dissimilar set sizes. We derive the reliability estimator when the random samples of strength $X \sim \mathrm{ILo}(\rho, \omega)$ and stress $Y \sim \mathrm{ILo}(\rho, \varphi)$ are observed from ERSS.

4.1. Estimator of $R = P[Y_{OSZ} < X_{OSZ}]$. Herein, we derive the reliability estimator when the observed data of strength X

and stress Y populations are drawn from the ERSS scheme with OSZ. Suppose that $\{X_{a(1)e}; a = 1, 2, \cdots, v-1\} \cup X_{m_1(v)e} \cup \{X_{a(m_1)e}, a = v, \cdots, (m_1-1)\}$ where $e = 1, 2, \cdots, q_x$ and $v = [(m_1+1)/2]$ are the ERSS scheme drawn from $X \sim \text{ILo}(\rho, \omega)$ with sample size $m_1 q_x$, where m_1 is the set size and q_x is the number of cycles. Let $X_{a(1)e}, X_{m_1(v)e}$, and $X_{a(m_1)e}$ are the smallest, median, and largest order statistics from the ath set of size m_1 of the eth cycle, respectively. The observed ERSS with OSZ (for one cycle) is presented in Table 1.

The pdfs of the smallest, median, and largest order statistics from the ath set of size m_1 of the eth cycle are defined, respectively, as follows.

$$
\begin{aligned}
h_{X_1}\left(x_{a(1)e}\right) &= m_1\left[1 - H\left(x_{a(1)e}\right)\right]^{m_1-1} h\left(x_{a(1)e}\right) \\
&= \frac{m_1 \rho \omega}{x_{a(1)e}^2}\left[1 - \left(1 + \frac{\rho}{x_{a(1)e}}\right)^{-\omega}\right]^{m_1-1} \\
&\quad \cdot \left(1 + \frac{\rho}{x_{a(1)e}}\right)^{-\omega-1}, x_{a(1)e} > 0,
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
h_{X_v}\left(x_{m_1(v)e}\right) &= \frac{m_1}{(v-1!)^2}\left[H\left(x_{m_1(v)e}\right)\right]^{v-1} \\
&\quad \cdot \left[1 - H\left(x_{m_1(v)e}\right)\right]^{v-1} h\left(x_{m_1(v)e}\right) \\
&= \frac{m_1! \rho \omega}{[(v-1)!]^2 x_{m_1(v)e}^2}\left(1 + \frac{\rho}{x_{m_1(v)e}}\right)^{-\omega v-1} \\
&\quad \cdot \left[1 - \left(1 + \frac{\rho}{x_{m_1(v)e}}\right)^{-\omega}\right]^{v-1}, x_{m_1(v)e} > 0,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
h_{X_{m_1}}\left(x_{a(m_1)e}\right) &= m_1\left[H\left(x_{a(m_1)e}\right)\right]^{m_1-1} h\left(x_{a(m_1)e}\right) \\
&= \frac{m_1 \rho \omega}{x_{a(m_1)e}^2}\left(1 + \frac{\rho}{x_{a(m_1)e}}\right)^{-\omega m_1-1}, x_{a(m_1)e} > 0.
\end{aligned}
\tag{16}
$$

Similarly, assume that $\{Y_{b(1)g}; b = 1, 2, \cdots, u-1\} \cup Y_{m_2(u)g} \cup \{Y_{b(m_2)g}, b = u, \cdots, m_2-1\}$, where $g = 1, 2, \cdots, q_y$ and $u = [(m_2+1)/2]$ are the ERSS drawn from $Y \sim \text{ILo}(\rho, \varphi)$ with a sample size $m_2 q_y$, where m_2 is the set size and q_y is the number of cycles. Let $Y_{b(1)g}, Y_{m_2(u)g}$, and $Y_{b(m_2)g}$ are the smallest, median, and largest order statistics from the bth set of size m_2 of the gth cycle, respectively. The pdfs of the smallest, median, and largest order statistics from the bth set of size m_2 of the gth cycle are defined, respectively, as follows:

$$
\begin{aligned}
h_{Y_1}\left(y_{b(1)g}\right) &= \frac{m_2 \rho \varphi}{y_{b(1)g}^2}\left[1 - \left(1 + \frac{\rho}{y_{b(1)g}}\right)^{-\varphi}\right]^{m_2-1} \\
&\quad \cdot \left(1 + \frac{\rho}{y_{b(1)g}}\right)^{-\varphi-1}, y_{b(1)g} > 0,
\end{aligned}
\tag{17}
$$

TABLE 1: The observed ERSS with OSZ.

					Observed
$X_{1(1)}$	...	$X_{1(v)}$	...	$X_{1(m_1)}$	$X_{1(1)}$
$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$X_{v-1(1)}$	...	$X_{v-1(v)}$	...	$X_{v-1(m_1)}$	$X_{v-1(1)}$
$X_{v(1)}$	...	$X_{v(v)}$	...	$X_{v(m_1)}$	$X_{v(m_1)}$
$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$X_{m_1-1(1)}$	...	$X_{m_1-1(v)}$	...	$X_{m_1-1(m_1)}$	$X_{m_1-1(m_1)}$
$X_{m_1(1)}$	...	$X_{m_1(v)}$	...	$X_{m_1(m_1)}$	$X_{m_1(v)}$

$$
\begin{aligned}
h_{Y_u}\left(y_{m_2(u)g}\right) &= \frac{m_2! \rho \varphi}{[(u-1)!]^2 y_{m_2(u)g}^2}\left(1 + \frac{\rho}{y_{m_2(u)g}}\right)^{-\varphi u-1} \\
&\quad \cdot \left[1 - \left(1 + \frac{\rho}{y_{m_2(u)g}}\right)^{-\varphi}\right]^{u-1}, y_{m_2(u)g} > 0,
\end{aligned}
\tag{18}
$$

$$
h_{Y_{m_2}}\left(y_{b(m_2)g}\right) = \frac{m_2 \rho \varphi}{y_{b(m_2)g}^2}\left(1 + \frac{\rho}{y_{b(m_2)g}}\right)^{-\varphi m_2-1}, y_{b(m_2)g} > 0.
\tag{19}
$$

The likelihood function, say ℓ_2, based on ERSS method with OSZ is given by the following.

$$
\begin{aligned}
\ell_2 &= \prod_{e=1}^{q_x}\left\{\left[\prod_{a=1}^{v-1} h_{X_1}\left(x_{a(1)e}\right)\right]\left[\prod_{a=v}^{m_1-1} h_{X_{m_1}}\left(x_{a(m_1)e}\right)\right] h_{X_v}\left(x_{m_1(v)e}\right)\right\} \\
&\quad \cdot \prod_{g=1}^{q_y}\left\{\left[\prod_{b=1}^{u-1} h_{Y_1}\left(y_{b(1)g}\right)\right] h_{Y_u}\left(y_{m_2(u)g}\right)\left[\prod_{b=u}^{m_2-1} h_{Y_{m_2}}\left(y_{b(m_2)g}\right)\right]\right\}.
\end{aligned}
\tag{20}
$$

The ML estimators of the parameters ω, φ, and ρ are the solutions of the following equations:

$$
\begin{aligned}
\frac{\partial \ell_2}{\partial \omega} &= \frac{n^\bullet}{\omega} + \sum_{e=1}^{q_x}\left[\sum_{a=1}^{v-1}\left\{\frac{(m_1-1)\ln Z_{a(1)e}}{\left(Z_{a(1)e}\right)^\omega - 1} - \ln Z_{a(1)e}\right\}\right. \\
&\quad - m_1 \sum_{a=v}^{m_1-1} \ln Z_{a(m_1)e} - v \ln\left(1 + \left(\frac{\rho}{x_{m_1(v)e}}\right)\right) \\
&\quad \left. + \frac{(v-1)\ln\left(1 + \left(\rho/x_{m_1(v)e}\right)\right)}{\left(1 + \left(\rho/x_{m_1(v)e}\right)\right)^\omega - 1}\right],
\end{aligned}
\tag{21}
$$

$$\frac{\partial \ell_2}{\partial \varphi} = \frac{m^{\bullet}}{\varphi} + \sum_{g=1}^{q_y} \left[\sum_{b=1}^{u-1} \left\{ \frac{(m_2-1)\ln D_{b(1)g}}{\left(D_{b(1)g}\right)^{\varphi}-1} - \ln D_{b(1)g} \right\} \right.$$
$$- m_2 \sum_{b=u}^{m_2-1} \ln D_{b(m_2)g} - u \ln \left(1 + \left(\frac{\rho}{y_{m_2(u)g}} \right) \right)$$
$$\left. + \frac{(u-1)\ln \left(1 + \left(\rho/y_{m_2(u)g} \right) \right)}{\left(1 + \left(\rho/y_{m_2(u)g} \right) \right)^{\varphi}-1} \right],$$

(22)

$$\frac{\partial \ell_2}{\partial \rho} = \frac{n^{\bullet}+m^{\bullet}}{\rho} + \sum_{e=1}^{q_x} \left[\sum_{a=1}^{v-1} \left[\frac{(m_1-1)\omega Z_{a(1)e}^{-\omega-1}}{x_{a(1)e}\left[1-Z_{a(1)e}^{-\omega}\right]} - \frac{(\omega+1)}{x_{a(1)e}+\rho} \right] \right.$$
$$- \sum_{a=v}^{m_1-1} \frac{(\omega m_1+1)}{x_{a(m_1)e}+\rho} \right] - \sum_{g=1}^{q_y} \sum_{b=u}^{m_2-1} \frac{(\varphi m_2+1)}{y_{b(m_2)g}+\rho}$$
$$\cdot \sum_{g=1}^{q_y} \left[\sum_{b=1}^{u-1} \left\{ \frac{(m_2-1)\varphi D_{b(1)g}^{-\varphi-1}}{y_{b(1)g}\left[1-D_{b(1)g}^{-\varphi}\right]} - \frac{(\varphi+1)}{y_{b(1)g}+\rho} \right\} \right]$$
$$- \sum_{e=1}^{q_x} \left[\frac{\omega v+1}{x_{m_1(v)e}+\rho} + \frac{(v-1)\omega\left(1+\left(\rho/x_{m_1(v)e}\right)\right)^{-\omega-1}}{x_{m_1(v)e}\left[1-\left(1+\left(\rho/x_{m_1(v)e}\right)\right)^{-\omega}\right]} \right]$$
$$- \sum_{g=1}^{q_y} \left[\frac{\varphi u+1}{y_{m_2(u)g}+\rho} + \frac{(u-1)\varphi\left(1+\left(\rho/y_{m_2(u)g}\right)\right)^{-\varphi-1}}{y_{m_2(u)g}\left[1-\left(1+\left(\rho/y_{m_2(u)g}\right)\right)^{-\varphi}\right]} \right],$$

(23)

where $Z_{a(\tau)e} = (1+\rho/x_{a(\tau)e})$, $D_{b(\tau_1)g} = (1+\rho/y_{b(\tau_1)g})$, $\tau = 1, m_1, \tau_1 = 1, m_2$. We obtain the parameter's estimator by solving numerically Equations (21)–(23) using an iterative technique. As a result, the S-S reliability estimator is produced from (3).

4.2. Estimator of $R = P[Y_{ESZ} < X_{ESZ}]$. Herein, we derive the reliability estimator when the observed data of strength X and stress Y distributions are drawn from the ERSS method with ESZ. Let $\{X_{a(1)e}; a = 1, 2, \cdots, c\} \cup \{X_{a(m_1)e} = c+1, \cdots, m_1\}$ where $e = 1, 2, \cdots, q_x$ and $c = [m_1/2]$ are the ERSS with ESZ drawn from $X \sim \text{ILo}(\rho, \omega)$ with sample size $m_1 q_x$. Let $X_{a(1)e}$ and $X_{a(m_1)e}$ are the smallest and largest order statistics from the ath set of size m_1 of the eth cycle, respectively. The observed ERSS with ESZ (for one cycle) is represented in Table 2.

The pdfs of $X_{a(1)e}$ and $X_{a(m_1)e}$ from the ath set of size m_1 of the eth cycle are defined in (14) and (16). Similarly, let $\{Y_{b(1)g}; b = 1, 2, \cdots, d\} \cup \{Y_{b(m_2)g}, b = d+1, \cdots, m_2\}$, where $g = 1, 2, \cdots, q_y$ and $d = [m_2/2]$ are the ERSS with ESZ drawn from $Y \sim \text{ILo}(\rho, \varphi)$ with sample size $m_2 q_y$. The pdfs of $Y_{b(1)g}$ and $Y_{b(m_2)g}$ from the bth set of size m_2 of the gth cycle are defined in (17) and (19). The likelihood

TABLE 2: The observed ERSS with ESZ.

					Observed
$X_{1(1)}$	...	$X_{1(c)}$	...	$X_{1(m_1)}$	$X_{1(1)}$
$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$X_{c(1)}$	...	$X_{c(c)}$	...	$X_{c(m_1)}$	$X_{c(1)}$
$X_{c+1(1)}$	...	$X_{c+1(c)}$	...	$X_{c+1(m_1)}$	$X_{c+1(m_1)}$
$\vdots$	...	$\vdots$	...	$\vdots$	$\vdots$
$X_{m_1(1)}$	...	$X_{m_1(c)}$	...	$X_{m_1(m_1)}$	$X_{m_1(m_1)}$

function, say ℓ_3, based on ERSS with ESZ, is given by the following:

$$\ell_3 = \prod_{e=1}^{q_x} \left[\prod_{a=1}^{c} h_{X_1}\left(x_{a(1)e}\right) \prod_{a=c+1}^{m_1} h_{X_{m_1}}\left(x_{a(m_1)e}\right) \right]$$
$$\cdot \prod_{g=1}^{q_y} \left[\prod_{b=1}^{d} h_{Y_1}\left(y_{b(1)g}\right) \prod_{b=d+1}^{m_2} h_{Y_{m_2}}\left(y_{b(m_2)g}\right) \right].$$

(24)

The ML estimators of ω, φ, and ρ are the solutions of the following likelihood equations:

$$\frac{\partial \ell_3}{\partial \omega} = \frac{n^{\bullet}}{\omega} + \sum_{e=1}^{q_x} \left[\sum_{a=1}^{c} \left(\frac{(m_1-1)\ln Z_{a(1)e}}{\left(Z_{a(1)e}\right)^{\omega}-1} - \ln Z_{a(1)e} \right) \right.$$
$$\left. - \sum_{a=c+1}^{m_1} m_1 \ln Z_{a(m_1)e} \right],$$

(25)

$$\frac{\partial \ell_3}{\partial \varphi} = \frac{m^{\bullet}}{\varphi} + \sum_{g=1}^{q_y} \left[\sum_{b=1}^{d} \left(\frac{(m_2-1)\ln D_{b(1)g}}{\left(D_{b(1)g}\right)^{\varphi}-1} - \ln D_{b(1)g} \right) \right.$$
$$\left. - \sum_{b=d+1}^{m_2} m_2 \ln D_{b(m_2)g} \right],$$

(26)

$$\frac{\partial \ell_3}{\partial \rho} = \frac{n^{\bullet}+m^{\bullet}}{\rho} + \sum_{e=1}^{q_x} \left[\sum_{a=1}^{c} \left(\frac{(m_1-1)\omega Z_{a(1)e}^{-\omega-1}}{x_{a(1)e}\left(1-Z_{a(1)e}^{-\omega}\right)} - \frac{\omega+1}{x_{a(1)e}+\rho} \right) \right.$$
$$\left. - \sum_{a=c+1}^{m_1} \frac{\omega m_1+1}{x_{a(m_1)e}+\rho} \right]$$
$$+ \sum_{g=1}^{q_y} \left[\sum_{b=1}^{d} \left(\frac{(m_2-1)\varphi D_{b(1)g}^{-\varphi-1}}{y_{b(1)g}\left(1-D_{b(1)g}^{-\varphi}\right)} - \frac{\varphi+1}{y_{b(1)g}+\rho} \right) \right.$$
$$\left. - \sum_{b=d+1}^{m_2} \frac{\varphi m_2+1}{y_{b(m_2)g}+\rho} \right].$$

(27)

Setting Equations (25)–(27) with zero and solving numerically, we obtain the ML estimators of ω, φ, and ρ. Consequently, the S-S reliability estimator is provided using (3).

4.3. Estimator of $R = P[Y_{ESZ} < X_{OSZ}]$. Here, we obtain the S-S reliability estimator when the observed samples of strength X are drawn from ERSS with OSZ, while observed samples of stress Y are drawn from ERSS with ESZ. Let $\{X_{a(1)e} ; a = 12, , \cdots, v-1\} \cup X_{m_1(v)e} \cup \{X_{a(m_1)e}, a = v, \cdots, m_1 - 1\}$ where $e = 1, 2, \cdots, q_x$ and $v = [(m_1+1)/2]$ are the ERSS drawn from $X \sim \text{ILo}(\rho, \omega)$ with sample size $m_1 q_x$. The pdfs of $X_{a(1)e}$, $X_{m_1(v)e}$, and $X_{a(m_1)e}$ are provided in Equations (14)–(16).

Suppose that $\{Y_{b(1)g} ; b = 1, 2, \cdots, d\} \cup \{Y_{b(m_2)g}, b = d+1, \cdots, m_2\}$, where $g = 1, 2, \cdots, qy$ and $d = [m_2/2]$ are the ERSS with ESZ drawn from $Y \sim \text{ILo}(\rho, \varphi)$ with sample size $m_2 q_y$, where the density function of $Y_{b(1)g}$ and $Y_{b(m_2)g}$ are obtained in Equations (17) and (19). Hence, the likelihood function, say ℓ_4, in this case, is given by the following:

$$
\ell_4 = \prod_{e=1}^{q_x} \prod_{a=1}^{v-1} h_{X_1}\left(x_{a(1)e}\right) \prod_{e=1}^{q_x} h_{X_v}\left(x_{m_1(v)e}\right)
$$
$$
\cdot \prod_{e=1}^{q_x} \prod_{a=v}^{m_1-1} h_{X_{m_1}}\left(x_{a(m_1)e}\right) \tag{28}
$$
$$
\cdot \prod_{g=1}^{q_y} \left[\prod_{b=1}^{d} h_{Y_1}\left(y_{b(1)g}\right) \prod_{b=d+1}^{m_2} h_{Y_{m_2}}\left(y_{b(m_2)g}\right) \right].
$$

The partial derivatives of ω and φ are provided in (21) and (26). The partial derivative of ρ is given by

$$
\frac{\partial \ell_4}{\partial \rho} = \frac{n^\bullet + m^\bullet}{\rho} + \sum_{e=1}^{q_x} \left[\sum_{a=1}^{v-1} \left\{ \frac{(m_1-1)\omega Z_{a(1)e}^{-\omega-1}}{x_{a(1)e}\left[1 - Z_{a(1)e}^{-\omega}\right]} - \frac{\omega+1}{x_{a(1)e}+\rho} \right\} \right.
$$
$$
\left. - \sum_{a=v}^{m_1-1} \frac{\omega m_1 + 1}{x_{a(m_1)e}+\rho} \right] - \sum_{g=1}^{q_y} \sum_{b=d}^{m_2} \frac{\varphi m_2 + 1}{y_{b(m_2)g}+\rho}
$$
$$
- \sum_{e=1}^{q_x} \left[\frac{\omega v + 1}{x_{m_1(v)e}+\rho} + \frac{(v-1)\omega\left(1 + \left(\rho/x_{m_1(v)e}\right)\right)^{-\omega-1}}{x_{m_1(v)e}\left[1 - \left(1 + \left(\rho/x_{m_1(v)e}\right)\right)^{-\omega}\right]} \right]
$$
$$
+ \sum_{g=1}^{q_y} \left[\sum_{b=1}^{d} \left(\frac{(m_2-1)\varphi D_{b(1)g}^{-\varphi-1}}{y_{b(1)g}\left(1 - D_{b(1)g}^{-\varphi}\right)} - \frac{\varphi+1}{y_{b(1)g}+\rho} \right) \right]. \tag{29}
$$

The parameter estimators of ω, φ, and ρ are the solutions of the Equations (21), (26), and (29), and after setting them to zero, the S-S reliability estimator is obtained consequentially from (3).

4.4. Estimator of $R = P[Y_{OSZ} < X_{ESZ}]$. Here, we obtain the S-S reliability estimator when the observed samples of strength X are drawn from ERSS with ESZ, while observed samples of stress Y are drawn from ERSS with OSZ.

Suppose that $\{X_{a(1)e} ; a = 1, 2, \cdots, c\} \cup \{X_{a(m_1)e}, a = c+1, \cdots, m_1\}$, where $e = 1, 2, \cdots, q_x$ and $c = [m_1/2]$ are the ERSS with ESZ drawn from $X \sim \text{ILo}(\rho, \omega)$ with sample size $m_1 q_x$, with pdfs (14) and (16). Let $\{Y_{b(1)g} ; b = 1, 2, \cdots, u-1\} \cup Y_{m_2(u)g} \cup \{Y_{b(m_2)g}, b = u, u+1, \cdots, m_2 - 1\}$, where $g = 1, 2, \cdots, q_y$ and $u = [(m_2+1)/2]$ are the ERSS drawn from $Y \sim \text{ILo}(\rho, \varphi)$ with sample size $m_2 q_y$, with pdfs (17)–(19). Hence, the likelihood function, say ℓ_5, in this case, is given by the following:

$$
\ell_5 = \prod_{e=1}^{q_x} \left[\prod_{a=1}^{c} h_{X_1}\left(x_{a(1)e}\right) \prod_{a=c+1}^{m_1} h_{X_{m_1}}\left(x_{a(m_1)e}\right) \right]
$$
$$
\cdot \prod_{g=1}^{q_y} \prod_{b=1}^{u-1} h_{Y_1}\left(y_{b(1)g}\right) \prod_{g=1}^{q_y} h_{Y_u}\left(y_{m_1(u)g}\right) \tag{30}
$$
$$
\cdot \prod_{g=1}^{q_y} \prod_{b=u}^{m_2-1} h_{Y_{m_2}}\left(y_{b(m_2)g}\right).
$$

The partial derivatives of φ and ω are provided in (22) and (25). The partial derivative of ρ is given by

$$
\frac{\partial \ell_5}{\partial \rho} = \frac{n^\bullet + m^\bullet}{\rho} + \sum_{e=1}^{q_x} \left[\sum_{a=1}^{c} \left(\frac{(m_1-1)\omega Z_{a(1)e}^{-\omega-1}}{x_{a(1)e}\left(1 - Z_{a(1)e}^{-\omega}\right)} - \frac{\omega+1}{x_{a(1)e}+\rho} \right) \right.
$$
$$
\left. - \sum_{a=c}^{m_1} \frac{\omega m_1 + 1}{x_{a(m_1)e}+\rho} \right] + \sum_{g=1}^{q_y} \sum_{b=1}^{u-1} \left(\frac{(m_2-1)\varphi D_{b(1)g}^{-\varphi-1}}{y_{b(1)g}\left[1 - D_{b(1)g}^{-\varphi}\right]} \right)
$$
$$
- \sum_{g=1}^{q_y} \left[\sum_{b=1}^{u-1} \frac{\varphi+1}{y_{b(1)g}+\rho} - \sum_{b=u}^{m_2-1} \frac{\varphi m_2 + 1}{y_{b(m_2)g}+\rho} \right]
$$
$$
- \sum_{g=1}^{q_y} \left[\frac{\varphi u + 1}{y_{m_2(u)g}+\rho} + \frac{(u-1)\varphi\left(1 + \left(\rho/y_{m_2(u)g}\right)\right)^{-\varphi-1}}{y_{m_2(u)g}\left[1 - \left(1 + \left(\rho/y_{m_2(u)g}\right)\right)^{-\varphi}\right]} \right]. \tag{31}
$$

The parameter estimators of φ, ω, and ρ are the solutions of Equations (22), (25), and (31) after setting them to zero. As a result, reliability estimator is obtained using (3).

5. Numerical Representation

This section introduces some simulations to assess how well the ML estimation of the S-S reliability function worked based on the proposed sampling scheme. A comparison is made between different estimates based on SRS, RSS, and ERSS methods. The following is a full description of the simulated experiment.

(i) Using inverse transformation, 1000 random samples are created from the strength $X \sim \text{ILo}(\rho, \omega)$ and stress $Y \sim \text{ILo}(\rho, \varphi)$ distributions

TABLE 3: Measures of $\widehat{R}$ for different sampling schemes at $R = 0.909$.

(m_1, m_2)	SRS			RSS			ERSS			Efficiency		
	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	RE_1	RE_2	RE_3
(2,2)	0.9177	0.0086	0.0026	0.9132	0.0041	0.0020	0.9021	0.0069	0.0012	1.2683	2.1667	1.7083
(2,3)	0.9141	0.0049	0.0020	0.9156	0.0065	0.0014	0.9347	0.0256	0.0010	1.4610	1.9808	1.3558
(3,2)	0.9065	0.0025	0.0022	0.9069	0.0027	0.0013	0.8227	0.0863	0.0129	1.7068	0.1750	0.1025
(3,3)	0.9088	0.0002	0.0018	0.9142	0.0051	0.0011	0.9342	0.0251	0.0011	1.6121	1.6549	1.0265
(3,4)	0.9099	0.0008	0.0013	0.9102	0.0011	0.0007	0.8897	0.0193	0.0010	1.8630	1.2477	0.6697
(4,3)	0.9105	0.0014	0.0012	0.9150	0.0059	0.0011	0.9244	0.0150	0.0005	1.1053	2.3774	2.1509
(4,4)	0.9132	0.0040	0.0013	0.9086	0.0004	0.0005	0.9079	0.0011	0.0004	2.6346	3.1860	1.2093
(5,5)	0.9095	0.0004	0.0014	0.9111	0.0020	0.0003	0.9357	0.0266	0.0009	3.7838	1.5217	0.4022
(6,6)	0.9051	0.0039	0.0008	0.9095	0.0004	0.0003	0.9092	0.0001	0.0002	2.2162	3.0370	1.3704
(7,7)	0.9085	0.0006	0.0008	0.9125	0.0034	0.0002	0.9361	0.0270	0.0008	3.0741	0.9651	0.3140

(ii) The parameter's values are chosen as $(\omega, \varphi) = (5, 0.5), (5, 1), (5, 2), (5, 3), \rho = 2$, and the true value for the system reliability R is determined as 0.909, 0,833, 0.714, and 0.625, respectively

(iii) The sample sizes are selected as $(n^\bullet, m^\bullet) = (10, 10), (10, 15), (15, 10), (15, 15), (15, 20), (20, 15), (20, 20), (25, 25), (30, 30), (30, 30)$ for SRS

(iv) The number of cycles is set to be $q_x = q_y = q = 5$, while the set sizes are selected as $(m_1, m_2) = (2, 2), (2, 3), (3, 2), (3, 3), (3, 4), (4, 3), (4, 4), (5, 5), (6, 6),$ and $(7, 7)$. As a result, the sample sizes for RSS and ERSS sampling designs are determined as $n^\bullet = m_1 q_x = m_1 q$ and $m^\bullet = m_2 q_y = m_2 q$

(v) A numerical technique is utilized to obtain the ML of parameters and consequently the reliability estimate using the three sampling strategies

(vi) The performance of the S-S reliability estimates for the three sampling strategies is evaluated using ABs, MSEs, and REs measures

(vii) The AB is defined as: $AB(R_\alpha) = |E(R_\alpha) - R|$, where $\alpha = (Y_{ERSS} < X_{ERSS}), (Y_{RSS} < X_{RSS}), (Y_{SRS} < X_{SRS})$.

(viii) Three REs of reliability estimates $\widehat{R}$ are provided and defined as follows:

$$RE_1 = \frac{MSE\left[\widehat{R}(Y_{SRS} < X_{SRS})\right]}{MSE\left[\widehat{R}(Y_{RSS} < X_{RSS})\right]},$$

$$RE_2 = \frac{MSE\left[\widehat{R}(Y_{SRS} < X_{SRS})\right]}{MSE\left[\widehat{R}(Y_{ERSS} < X_{ERSS})\right]}, \quad (32)$$

$$RE_3 = \frac{MSE\left[\widehat{R}(Y_{RSS} < X_{RSS})\right]}{MSE\left[\widehat{R}(Y_{ERSS} < X_{ERSS})\right]}.$$

(ix) Tables 3–6 describe the reliability estimates $\widehat{R}$, ABs, and MSEs based on SRS, RSS, and ERSS schemes. The REs of $\widehat{R}$ based on ERSS and RSS with respect to SRS and RSS for various sample sizes are presented in Tables 3–6

Tables 3–6 and Figures 1–6 show the following numerical results:

(i) The reliability estimates via RSS are more efficient than the corresponding based on SRS based in the same number of measured units

(ii) With the exception of $(m_1, m_2) = (3, 2)$, $\widehat{R}$ obtained by ERSS are more efficient than those obtained through SRS

(iii) Except for $(m_1, m_2) = (2, 2), (2, 3), (3, 3), (4, 3)$, the MSEs of $\widehat{R}$ based on RSS technique are smaller than the corresponding via ERSS scheme at actual value = 0.909, as shown in Table 3

(iv) Except for $(m_1, m_2) = (2, 3), (4, 3), (7, 7)$, the MSEs of $\widehat{R}$ based on RSS scheme are more efficient than the corresponding via ERSS at true value $R = 0.833$ (Table 4)

(v) At true value $R = 0.714$, the MSEs of $\widehat{R}$ based on RSS scheme are more efficient than the corresponding via ERSS except for $(m_1, m_2) = (2, 2), (5, 5)$ as seen in Table 5

(vi) At actual value $R = 0.625$, the MSEs of $\widehat{R}$ based on RSS scheme are more efficient than the corresponding ERSS, except for $(m_1, m_2) = (2, 2), (3, 3), (4, 3)$ (see Table 6)

(vii) The AB of RSS is smaller than SRS and ERSS in most of the cases

(viii) Expect at true value $R = 0.909$ for $(m_1, m_2) = (2, 3)$, the MSE of $\widehat{R}$ gets the largest value via SRS design and smallest values via RSS scheme (see Figure 1)

(ix) Expect at true value $R = 0.833$, the MSE of $\widehat{R}$ based on ERSS scheme is smaller than those under RSS and SRS methods at $(m_1, m_2) = (2, 2)$ (Figure 2)

TABLE 4: Measures of $\widehat{R}$ for different sampling schemes at $R = 0.833$.

(m_1, m_2)	SRS			RSS			ERSS			Efficiency		
	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	RE_1	RE_2	RE_3
(2,2)	0.8467	0.0133	0.0056	0.8506	0.0172	0.0043	0.8106	0.0227	0.0053	1.2910	1.0607	0.8216
(2,3)	0.8356	0.0023	0.0048	0.8368	0.0035	0.0024	0.8243	0.0090	0.0024	1.9959	1.9876	0.9959
(3,2)	0.8415	0.0082	0.0043	0.8401	0.0067	0.0026	0.7375	0.0958	0.0131	1.6500	0.3277	0.1986
(3,3)	0.8316	0.0018	0.0036	0.8355	0.0022	0.0022	0.8609	0.0276	0.0026	1.6484	1.3992	0.8488
(3,4)	0.8281	0.0052	0.0044	0.8340	0.0006	0.0023	0.8058	0.0275	0.0028	1.9295	1.5755	0.8165
(4,3)	0.8436	0.0102	0.0032	0.8341	0.0007	0.0019	0.8527	0.0193	0.0017	1.6545	1.8810	1.1369
(4,4)	0.8335	0.0002	0.0028	0.8334	0.0001	0.0014	0.8652	0.0318	0.0018	2.0072	1.5475	0.7709
(5,5)	0.8345	0.0011	0.0023	0.8384	0.0051	0.0010	0.8471	0.0138	0.0018	2.4421	1.2747	0.5220
(6,6)	0.8380	0.0047	0.0014	0.8318	0.0015	0.0005	0.8332	0.0001	0.0007	2.8600	2.1343	0.7463
(7,7)	0.8374	0.0041	0.0014	0.8360	0.0026	0.0005	0.8311	0.0023	0.0005	2.5000	2.8723	1.1489

TABLE 5: Measures of $\widehat{R}$ for different sampling schemes at $R = 0.714$.

(m_1, m_2)	SRS			RSS			ERSS			Efficiency		
	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	RE_1	RE_2	RE_3
(2,2)	0.7105	0.0038	0.0094	0.7153	0.0010	0.0082	0.6921	0.0222	0.0067	1.1569	1.4006	1.2107
(2,3)	0.7215	0.0072	0.0088	0.7144	0.0001	0.0039	0.7419	0.0276	0.0043	2.2545	2.0301	0.9005
(3,2)	0.7189	0.0047	0.0089	0.7156	0.0013	0.0055	0.6229	0.0914	0.0144	1.6148	0.6223	0.3854
(3,3)	0.7063	0.0080	0.0082	0.7230	0.0087	0.0035	0.6596	0.0547	0.0064	2.3671	1.2777	0.5398
(3,4)	0.7167	0.0025	0.0052	0.7192	0.0050	0.0025	0.6785	0.0358	0.0045	2.1215	1.1542	0.5441
(4,3)	0.7137	0.0005	0.0068	0.7168	0.0025	0.0034	0.6526	0.0617	0.0059	2.0208	1.1601	0.5741
(4,4)	0.7186	0.0043	0.0044	0.7201	0.0058	0.0021	0.7044	0.0099	0.0031	2.0909	1.3917	0.6656
(5,5)	0.7259	0.0116	0.0044	0.7136	0.0007	0.0015	0.7122	0.0021	0.0014	2.9329	3.1898	1.0876
(6,6)	0.7175	0.0032	0.0033	0.7098	0.0045	0.0011	0.7071	0.0072	0.0016	3.0370	2.0000	0.6585
(7,7)	0.7014	0.0129	0.0029	0.7109	0.0034	0.0008	0.7115	0.0028	0.0011	3.5556	2.6667	0.7500

TABLE 6: Measures of $\widehat{R}$ for different sampling schemes at $R = 0.625$.

(m_1, m_2)	SRS			RSS			ERSS			Efficiency		
	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	$\widehat{R}$	AB	MSE	RE_1	RE_2	RE_3
(2,2)	0.6310	0.0060	0.0106	0.6311	0.0061	0.0080	0.6041	0.0209	0.0071	1.3167	1.4915	1.1328
(2,3)	0.6222	0.0028	0.0083	0.6252	0.0002	0.0058	0.6810	0.0560	0.0076	1.4251	1.0895	0.7645
(3,2)	0.6095	0.0156	0.0112	0.6225	0.0025	0.0070	0.5280	0.0970	0.0146	1.5852	0.7623	0.4809
(3,3)	0.6419	0.0169	0.0093	0.6301	0.0051	0.0041	0.6029	0.0221	0.0039	2.2451	2.3657	1.0537
(3,4)	0.6371	0.0121	0.0065	0.6305	0.0055	0.0025	0.5825	0.0426	0.0061	2.5754	1.0587	0.4111
(4,3)	0.6283	0.0033	0.0076	0.6278	0.0028	0.0032	0.6093	0.0157	0.0021	2.3634	3.7122	1.5707
(4,4)	0.6117	0.0133	0.0057	0.6283	0.0033	0.0028	0.5638	0.0612	0.0051	2.0106	1.1135	0.5538
(5,5)	0.6290	0.0040	0.0055	0.6290	0.0040	0.0016	0.6429	0.0179	0.0026	3.4313	2.0954	0.6107
(6,6)	0.6214	0.0036	0.0050	0.6270	0.0020	0.0011	0.6354	0.0104	0.0017	4.5596	2.9064	0.6374
(7,7)	0.6297	0.0047	0.0028	0.6211	0.0039	0.0009	0.6422	0.0172	0.0020	3.0769	1.3861	0.4505

(x) The MSE of $\widehat{R}$ based on ERSS obtains the fewest values compared to the others under RSS and SRS at true value $R = 0.714$ and $(m_1, m_2) = (5, 5)$ (see Figure 3)

(xi) Figure 4 indicates that, at real value $= 0.909$, the MSE of $\widehat{R}$ based on the ERSS scheme is smaller than the comparable via RSS and SRS

(xii) Figures 3 and 4 indicate that the MSE of $\widehat{R}$ decreases as the true value of R increases

(xiii) For $(m_1, m_2) = (2, 2)$ the RE_2 of $\widehat{R}$ based on ERSS scheme is more efficient than those under RSS and SRS except at true value $R = 0.833$, as shown in Figure 5
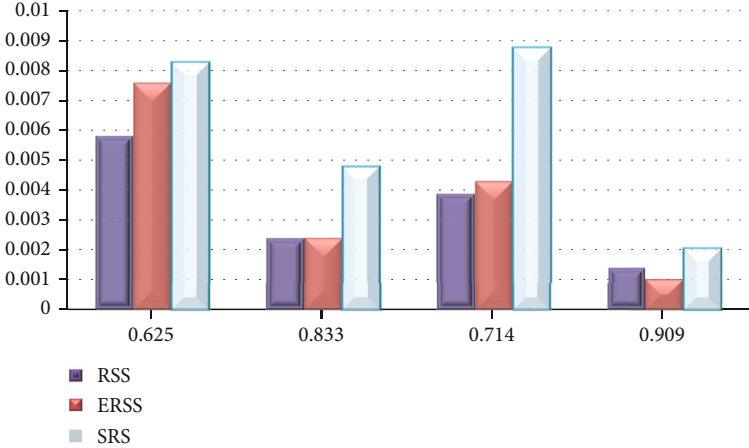
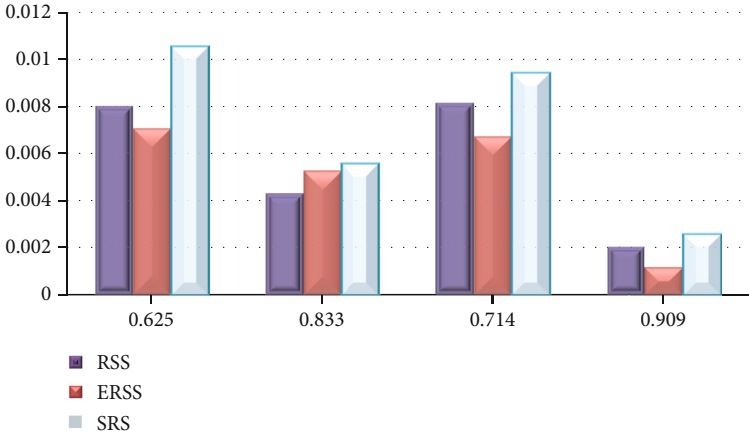FIGURE 1: MSE of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (2, 3)$.

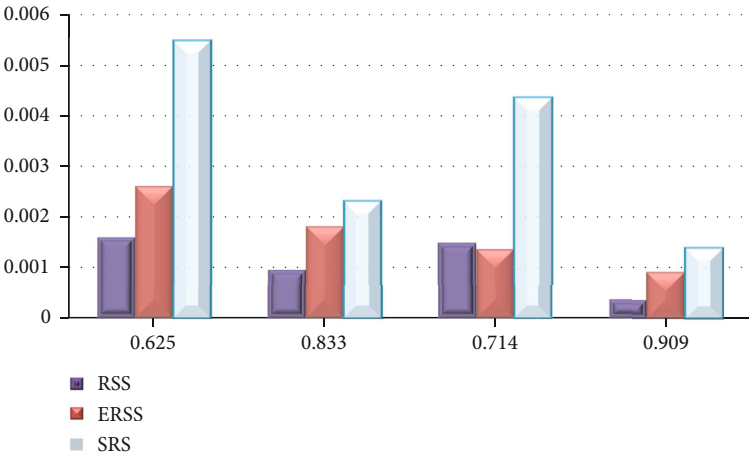FIGURE 2: MSE of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (2, 2)$.

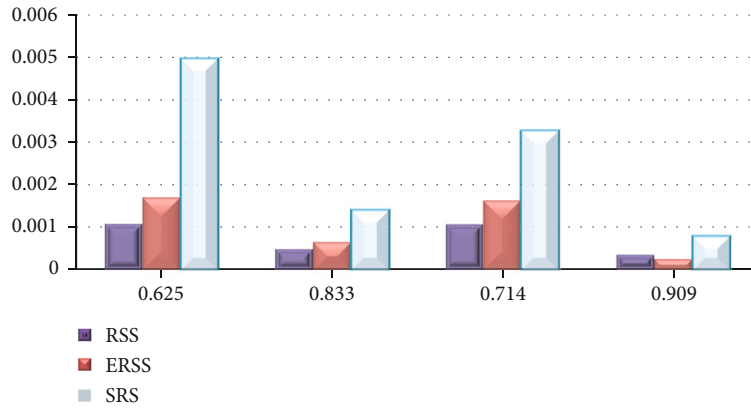FIGURE 3: MSE of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (5, 5)$.

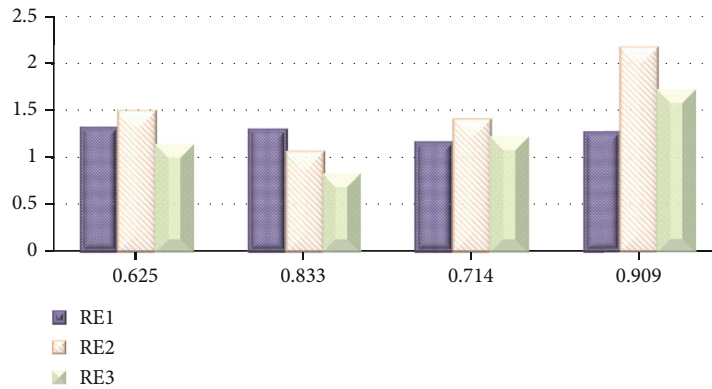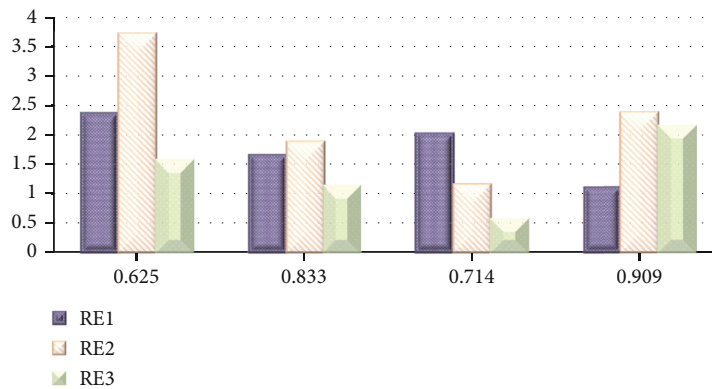Figure 4: MSE of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (6, 6)$.

Figure 5: Relative efficiency of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (2, 2)$.

Figure 6: Relative efficiency of $\widehat{R}$ for different sampling schemes at $(m_1, m_2) = (4, 3)$.

(xiv) Figure 6 illustrates that at $(m_1, m_2) = (4, 3)$, the RE_2 of the ERSS scheme is more efficient than those via RSS and SRS schemes with the exception of true value $R = 0.714$

(xv) The MSEs of the S-S reliability estimate in all schemes decrease as the actual value of R increases in most of the cases (see Figures 1–6).

6. Conclusions

This article tackles the estimation of the S-S reliability $R = P[Y < X]$ when the strength X and stress Y are independent inverse Lomax distributed random variables. Maximum likelihood estimators of R are computed using the SRS, RSS, and ERSS schemes. The reliability estimator is computed in four situations using ERSS design. Simulation

research is conducted to evaluate the performance of the proposed estimates. From the simulation outcomes, it is observed that the MSEs of reliability estimates based on SRS data are bigger than the comparable based on RSS and ERSS data, respectively. In most cases, at $R = 0.909$, the MSEs of reliability estimates under ERSS are the shortest when compared to similar estimators based on RSS and SRS data. The efficiency of all estimates improves as the actual value of reliability increases in almost all cases. This study showed that the reliability estimates based on RSS are more efficient than those based on ERSS and SRS. For most actual values of R, the reliability estimate via the ERSS technique is more efficient than those under RSS and SRS for small even set sizes. In some cases, estimates of reliability obtained by ERSS are more efficient than those obtained through RSS and SRS designs. In a future work, one may consider the problem of estimating R based on double extreme ranked set sampling [31], modified robust extreme ranked set sampling [32], stratified quartile ranked set sampling [33], and multistage percentile and quartile ranked set samples methods [34, 35].

Data Availability

There is no data is included in the paper.

Conflicts of Interest

The authors declare that they have no conflicts of interest to report regarding the present study.

Acknowledgments

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