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Why Are There as Many Elements in the Cantor Set as There Are Real Numbers?

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Abstract

There are many important concepts in linear algebra, such as linear correlation and linear independence, eigenvalues and eigenvectors, and so on. The article provides a graphical explanation of how to distinguish between the concepts of linear correlation and linear independence. The conclusion points out that linear independence means that there are no two (base) vectors with the same direction in a vector graph; otherwise, it is a linear correlation.

Keywords

Cantor Ternary Set, Linear Independence, Vector, Linear Algebra

In mathematics, the Cantor set, introduced by German mathematician Georg Cantor in 1883 (but discovered by Henry John Stephen Smith in 1875), is a set of points located on a line segment, with many significant and profound properties. By considering this set, Cantor and other mathematicians laid the foundation of modern point set topology. Although Cantor himself defined this set in a general and abstract way, the most common construction is Cantor ternary set which is obtained by removing the middle third of a line segment [1] [2] [3] [4] [5].

Take a straight line segment with a length of 1, divide it into three equal parts, remove the middle segment, leave the remaining two segments, and then divide the remaining two segments into three equal parts, remove the middle segment, and leave the shorter four segments… Continue this operation until infinity. As the number of segments formed during the continuous segmentation and discarding process increases, the length decreases, and at the limit, a discrete set of points is obtained, it is called the Cantor point set, denoted as P.

The length of the limit graph called the Cantor point set tends to 0, and the number of line segments tends to infinity, which is actually equivalent to a point set. After n operations [2], Side length $r = (1/3)^n$,

Number of edges $N(r) = 2^n$,

According to the formula $D = \ln N(r)/\ln(1/r)$, $D = \ln 2/\ln 3 = 0.631$.

So the fractional dimension of the Cantor point set is 0.631.

Why are there as many elements in the cantor set (in **Figure 1**) as there are real numbers? To illustrate this point, we represent all numbers between 0 and 1 using a ternary decimal. If these numbers are evenly divided into three sections, they are exactly the numbers between 0 and 0.1, between 0.1 and 0.2, and between 0.2 and 1. The number in the middle section that was dug out is exactly the number with the first digit after the decimal point being 1, while the remaining numbers are all numbers with the first digit after the decimal point being 0 or 2. The next step divides the left 1/3 into three segments, and they will be numbers between 0.00 and 0.01, numbers between 0.01 and 0.02, numbers between 0.02 and 0.10, and so on. We will dig out the middle one-third of each interval, which means all numbers with a second decimal place of 1 after the decimal point, and the remaining numbers will be those numbers with a second decimal place of 0 or 2 after the decimal point… Continuously operating in this way, the numbers left in the Cantor set are exactly those ternary decimals composed only of 0 and 2!

It should be noted that the leftmost column starts from the second row and can be represented by numbers 0.0, 0.00, 0.00, and so on; the rightmost column starts from the second row and can be represented by the numbers 0.2, 0.22, 0.222, and so on. Adding other columns, all the elements of the cantor set are filled with the entire digital space composed of 0 and 2. For example, 1/4 of the decimal representation is 0.020202, indicating that 1/4 indeed belongs to the Cantor set [6] [7].

Due to 1/4 being less than 1/3 = 3/9, the 1/3 part on the left side of the second line is located; at the same time, it is greater than 2/9, so it is in the second green part of the third line; and so on.

In addition, although one-third of the ternary decimals are 0.1 and one-third of the ternary decimals are 0.01, we can also rewrite them as infinite recurring decimals 0.02222… and 0.002222… (this principle is the same as 0.999… = 1), so they are also in the Cantor set.

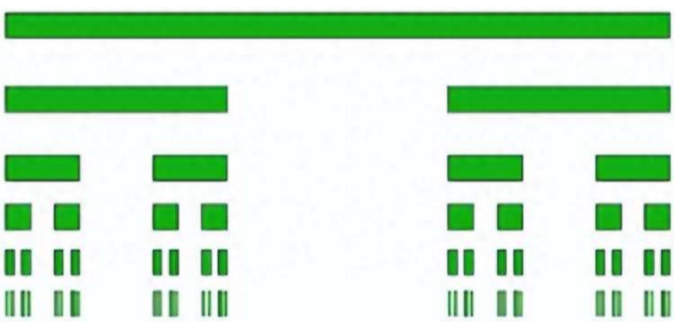

Figure 1. Cantor set.

Next, we can easily correspond the numbers in the Cantor set to all real numbers in the [0, 1] interval one by one. For any number in the Cantor set, first convert it to a ternary decimal, convert all the digits 2 in the decimal expansion to the digit 1, treat the new decimal as a binary decimal, and convert it back to the decimal system. It is a real number between [0, 1].

Similarly, for each real number in the [0, 1] interval, first write it as a binary decimal, then rewrite all 1 as 2, and treat it as a ternary decimal, making it a number in the Cantor set.

Therefore, there is a one-to-one correspondence between the numbers in the Cantor set and the real numbers in the entire [0, 1] interval, as there are equally many numbers in both sets!

The "total length" of the Cantor set is 0, but it contains as many elements as the real number interval. These singular properties play an important role in various mathematical branches such as measure theory, real analysis, and fractal theory.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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