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On Hyperbolic Generalized Woodall Numbers

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Authors' contributions

This work was carried out in collaboration between both authors. Both authors read and approved the final manuscript.

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ABSTRACT

In this study, we introduce the generalized hyperbolic Woodall numbers. As special cases, we study with hyperbolic Woodall, hyperbolic modified Woodall, hyperbolic Cullen numbers and hyperbolic modified Cullen numbers. We present Binet's formulas, generating functions and the summation formulas for these numbers. Besides, we give Catalan's and Cassini's identities and present matrices related to these sequences.

Keywords: Woodall numbers; cullen numbers; hyperbolic numbers; hyperbolic Woodall numbers; hyperbolic Cullen numbers.

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1 INTRODUCTION

In this section, we give some information which we need about the definition and properties of Woodall numbers.

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1.1 Woodall Numbers

The generalized Woodall sequence $\{W_n\}_{n\geq 0} = \{W_n(W_0, W_1, W_2, 5, -8, 4)\}_{n\geq 0}$ is defined by the third-order recurrence relations

$$W_n = 5W_{n-1} - 8W_{n-2} + 4W_{n-3} \tag{1.1}$$

with the initial values W_0, W_1, W_2 not all being zero. The sequence $\{W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$W_{-n} = 2W_{-(n-1)} - \frac{5}{4}W_{-(n-2)} + \frac{1}{4}W_{-(n-3)}$$

for $n = 1, 2, 3, ...$ Therefore, recurrence (1.1) holds for all integer n.

In the following theorem, we give Binet formula of generalized Woodall numbers.

Theorem 1.1. *[53, Theorem 1.1] Binet formula of generalized Woodall numbers can be given as*

$$W_n = (A_1 + A_2 n) \times 2^n + A_3$$

where

$$\begin{array}{rcl} A_1 & = & -W_2 + 4W_1 - 3W_0, \\ A_2 & = & \dfrac{W_2 - 3W_1 + 2W_0}{2}, \\ A_3 & = & W_2 - 4W_1 + 4W_0, \end{array}$$

that is,

$$W_n = ((-W_2 + 4W_1 - 3W_0) + \frac{W_2 - 3W_1 + 2W_0}{2} n) \times 2^n + (W_2 - 4W_1 + 4W_0). \tag{1.2}$$

Here, α, β and γ are the roots of the cubic equation

$$x^3 - 5x^2 + 8x - 4 = (x-2)^2(x-1) = 0,$$

where $\alpha = \beta = 2$, $\gamma = 1$.

Now, the first few generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 1.

Table 1. A few generalized Woodall numbers

n	W_n	W_{-n}
0	W_0	W_0
1	W_1	$\frac{1}{4}(8W_0 - 5W_1 + W_2)$
2	W_2	$\frac{1}{4}(11W_0 - 9W_1 + 2W_2)$
3	$4W_0 - 8W_1 + 5W_2$	$\frac{1}{16}(52W_0 - 47W_1 + 11W_2)$
4	$20W_0 - 36W_1 + 17W_2$	$\frac{1}{16}(57W_0 - 54W_1 + 13W_2)$
5	$68W_0 - 116W_1 + 49W_2$	$\frac{1}{64}(240W_0 - 233W_1 + 57W_2)$

Now, we define four specific cases of the sequence $\{W_n\}$.

The Woodall numbers $\{R_n\}$, sometimes called Riesel numbers, and also called Cullen numbers of the second kind, are numbers of the form

$$R_n = n \times 2^n - 1.$$

The first few Woodall numbers are:

$$1, 7, 23, 63, 159, 383, 895, 2047, 4607, 10239, 22527, 49151, 106495, 229375, 491519, 1048575, \ldots$$

(sequence A003261 in the OEIS [47]). Woodall numbers were first studied by Allan J. C. Cunningham and H. J. Woodall in [13] in 1917, inspired by James Cullen's earlier study of the similarly-defined Cullen numbers.

The Cullen numbers $\{C_n\}$ are numbers of the form

$$C_n = n \times 2^n + 1.$$

The first few Cullen numbers are:

$$1, 3, 9, 25, 65, 161, 385, 897, 2049, 4609, 10241, 22529, 49153, 106497, 229377, 491521, ...$$

(sequence A002064 in the OEIS). Woodall and Cullen sequences have been studied by many authors and more detail can be found in the extensive literature dedicated to these sequences, see [5,6,13,24,26,29,32,37,38,39,40]. Note that $\{R_n\}$ and $\{C_n\}$ hold the following relations:

$$\begin{aligned} R_n &= 4R_{n-1} - 4R_{n-2} - 1, \\ C_n &= 4C_{n-1} - 4C_{n-2} + 1. \end{aligned}$$

Note also that the sequences $\{R_n\}$ and $\{C_n\}$ satisfy the following third order linear recurrences:

$$\begin{aligned} R_n &= 5R_{n-1} - 8R_{n-2} + 4R_{n-3}, \quad R_0 = -1, R_1 = 1, R_2 = 7, \\ C_n &= 5C_{n-1} - 8C_{n-2} + 4C_{n-3}, \quad C_0 = 1, C_1 = 3, C_2 = 9. \end{aligned}$$

If we set $G_0 = 0$, $G_1 = 1$, $G_2 = 5$ then $\{G_n\}$ is the well-known modified Woodall sequence, if we set $H_0 = 3$, $H_1 = 5$, $H_2 = 9$ then $\{H_n\}$ is the well-known modified Cullen sequence. In other words, modified Woodall sequence $\{G_n\}_{n\geq 0}$ and modified Cullen sequence $\{H_n\}_{n\geq 0}$ are defined by the third-order recurrence relations

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3}, \quad G_0 = 0, G_1 = 1, G_2 = 5, \tag{1.3}$$

$$H_n = 5H_{n-1} - 8H_{n-2} + 4H_{n-3}, \quad H_0 = 3, H_1 = 5, H_2 = 9, \tag{1.4}$$

The sequences $\{G_n\}_{n\geq 0}$, $\{H_n\}_{n\geq 0}$, $\{R_n\}_{n\geq 0}$ and $\{C_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\begin{aligned} G_{-n} &= 2G_{-(n-1)} - \frac{5}{4}G_{-(n-2)} + \frac{1}{4}G_{-(n-3)}, \\ H_{-n} &= 2H_{-(n-1)} - \frac{5}{4}H_{-(n-2)} + \frac{1}{4}H_{-(n-3)}, \\ R_{-n} &= 2R_{-(n-1)} - \frac{5}{4}R_{-(n-2)} + \frac{1}{4}R_{-(n-3)}, \\ C_{-n} &= 2C_{-(n-1)} - \frac{5}{4}C_{-(n-2)} + \frac{1}{4}C_{-(n-3)}, \end{aligned}$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrences (1.3) and (1.4) hold for all integer n.

Using the initial conditions in (1.2), Binet's formula of modified Woodall, modified Cullen, Woodall and Cullen sequences are

$$\begin{aligned} G_n &= (n-1)\,2^n + 1, \\ H_n &= 2^{n+1} + 1, \\ R_n &= n \times 2^n - 1, \\ C_n &= n \times 2^n + 1. \end{aligned}$$

Now, we give the generating function and the Cassini identity for generalized Woodall numbers.

The generating function for generalized Woodall numbers is

$$\sum_{n=0}^{\infty} W_n x^n = \frac{W_0 + (W_1 - 5W_0)x + (W_2 - 5W_1 + 8W_0)x^2}{1 - 5x + 8x^2 - 4x^3}. \tag{1.5}$$

The Cassini identity for generalized Woodall numbers is

$$\begin{aligned}
W_{n+1}W_{n-1} - W_n^2 &= \frac{1}{4}2^n(A + B2^n + Cn). \\
A &= 4W_1^2 + W_2^2 - 4W_0W_1 + 4W_0W_2 - 5W_1W_2. \\
B &= -4W_0^2 - 9W_1^2 - W_2^2 + 12W_0W_1 - 4W_0W_2 + 6W_1W_2. \\
C &= 8W_0^2 + 12W_1^2 + W_2^2 - 20W_0W_1 + 6W_0W_2 - 7W_1W_2.
\end{aligned}$$

For further information about generalized Woodall numbers, see [53].

Next, we give some information about special numbers. In 1989, I. Kantor is worked the hypercomplex numbers systems, [31]. This numbers systems are extensions of real numbers. Some commutative some of hypercomplex number systems are defined below.

Complex numbers are

$$\mathbb{C} = \{z = a + ib : a, b \in \mathbb{R}, i^2 = -1\},$$

hyperbolic (double, split-complex) numbers [48] are

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R}, j^2 = 1, j \neq \pm 1\}$$

and dual numbers [20] are

$$\mathbb{D} = \{d = a + \varepsilon b : a, b \in \mathbb{R}, \varepsilon^2 = 0, \varepsilon \neq 0\}.$$

One of the non-commutative examples of hypercomplex number systems are quaternions, [28],

$$\mathbb{H}_{\mathbb{Q}} = \{q = a_0 + ia_1 + ja_2 + ka_3 : a_0, a_1, a_2, a_3 \in \mathbb{R},\ i^2 = j^2 = k^2 = ijk = -1\},$$

octonions [3] and sedenions [51]. The algebras $\mathbb{C}$ (complex numbers), $\mathbb{H}_{\mathbb{Q}}$ (quaternions), $\mathbb{O}$ (octonions) and $\mathbb{S}$ (sedenions) are real algebras obtained from the real numbers $\mathbb{R}$ by a doubling procedure called the Cayley-Dickson Process. This doubling process can be extended beyond the sedenions to form what are known as the 2^n-ions (see for example [7], [30], [41]).

Quaternions were invented by Irish mathematician W. R. Hamilton (1805-1865) [28] as an extension to the complex numbers. Hyperbolic numbers with complex coefficients are introduced by J. Cockle in 1848, [12].

Now, we will give some information related to hyperbolic numbers. We present hyperbolic numbers as follows:

$$\mathbb{H} = \{h = a + jb : a, b \in \mathbb{R},\ j^2 = 1,\ j \neq \pm 1\}.$$

The base elements $\{1, j\}$ of hyperbolic numbers satisfy the following properties (commutative multiplications):

$$1.j = j,\ j^2 = j.j = 1$$

where j symbolizes the hyperbolic unit ($j^2 = 1$).

The multiplication of two hyperbolic numbers $m = a_0 + ja_1$ and $n = b_0 + jb_1$ is

$$mn = a_0b_0 + a_1b_1 + j(a_0b_1 + a_1b_0).$$

Sum of two hyperbolic numbers $m = a_0 + ja_1$ and $n = b_0 + jb_1$ is

$$m + n = a_0 + b_0 + j(a_1 + b_1).$$

Now, we give details about hyperbolic and some information related to hyperbolic sequences from the literature.

- Richter, [44] worked On Hyperbolic Complex Numbers.
- Gürses, Şentürk and Yüce, [25] studied A Study on Dual-Generalized Complex and Hyperbolic-Generalized Complex numbers.

- Cockle [12] worked the Hyperbolic numbers with complex coefficients.
- Aydın, [1] worked hyperbolic Fibonacci numbers given by

$$\widetilde{F}_n = F_n + hF_{n+1},\ (h^2 = 1)$$

where Fibonacci numbers, respectively, given by $F_n = F_{n-1} + F_{n-2}$ with the initial condition $F_1 = F_2 = 1$, $(n \geq 3)$.

- Dikmen, [17] worked hyperbolic Jacobsthal numbers given by

$$\widehat{J}_n = J_n + hJ_{n+1}\ ,\ (h^2 = 1)$$

where Jacobsthal numbers, respectively, given by $J_n = J_{n-1} + 2J_{n-2}$, $J_0 = 0$, $J_1 = 1$.

- Taş, [74] worked on hyperbolic Jacobsthal-Lucas sequence given by

$$HJ_n = J_n + hJ_{n+1}, (h^2 = 1)$$

where Jacobsthal-Lucas numbers, respectively, given by $J_{n+2} = J_{n+1} + 2J_n$, with the initial condition $J_0 = 2$, $J_1 = 1$.

- Soykan and Taşdemir, [57] worked on hyperbolic generalized Jacobsthal numbers given by

$$\widetilde{V}_n = V_n + hV_{n+1}\ ,\ (h^2 = 1)$$

where generalized Jacobsthal numbers are given by $V_n = V_{n-1} + 2V_{n-2}$, $V_0 = a$, $V_1 = b$ $(n \geq 2)$ with the initial values V_0, V_1 not all being zero.

- Dişkaya, Menken, Catarino, [19] worked on hyperbolic Leonardo and hyperbolic Francois quaternions given by

$$\begin{array}{rcl} H\mathcal{L}_n &=& \mathcal{L}_n e_0 + \mathcal{L}_{n+1} e_1 + \mathcal{L}_{n+2} e_2 + \mathcal{L}_{n+3} e_3, \\ H\mathcal{F}_n &=& \mathcal{F}_n e_0 + \mathcal{F}_{n+1} e_1 + \mathcal{F}_{n+2} e_2 + \mathcal{F}_{n+3} e_3 \end{array}$$

where Francois and Leonardo numbers, respectively, given by $\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + 1$, with the initial condition $\mathcal{F}_0 = 2$, $\mathcal{F}_1 = 1$ and $\mathcal{L}_{n+2} = \mathcal{L}_{n+1} + \mathcal{L}_n$, with the initial condition $\mathcal{L}_0 = 1$, $\mathcal{L}_1 = 1$.

- Dikmen and Altınsoy, [18] worked on third order hyperbolic Jacobsthal numbers given by

$$\begin{array}{rcl} \widehat{J}_n^{(3)} &=& J_n^{(3)} + hJ_{n+1}^{(3)}, \\ \widehat{j}_n^{(3)} &=& j_n^{(3)} + hj_{n+1}^{(3)} \end{array}$$

where Jacobsthal numbers, respectively, given by $J_n^{(3)} = J_{n-1}^{(3)} + J_{n-2}^{(3)} + 2J_{n-3}^{(3)}$, $J_0^{(3)} = 0$, $J_1^{(3)} = 1$, $J_2^{(3)} = 1$, $j_n^{(3)} = j_{n-1}^{(3)} + j_{n-2}^{(3)} + 2j_{n-3}^{(3)}$, $j_0^{(3)} = 2$, $j_1^{(3)} = 1$, $j_2^{(3)} = 5$.

Next section, we present the hyperbolic generalized Woodall numbers and their generating functions and Binet's formulas.

2 HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we define hyperbolic generalized Woodall numbers and present generating functions and Binet's formulas for these numbers.

We now define hyperbolic generalized Woodall numbers over $\mathbb{H}$. The nth hyperbolic generalized Woodall number is

$$\mathcal{H}W_n = W_n + jW_{n+1}. \tag{2.1}$$

with the initial values $\mathcal{H}W_0, \mathcal{H}W_1, \mathcal{H}W_2$. (2.1) can be written to negative subscripts by defining,

$$\mathcal{H}W_{-n} = W_{-n} + jW_{-n+1}.$$

so identity (2.1) holds for all integers n.

The special cases of the nth dual hyperbolic generalized Woodall numbers are given as

$$\begin{aligned}
\mathcal{H}G_n &= G_n + jG_{n+1}, \\
\mathcal{H}H_n &= H_n + jH_{n+1}, \\
\mathcal{H}R_n &= R_n + jR_{n+1}, \\
\mathcal{H}C_n &= C_n + jC_{n+1}.
\end{aligned}$$

Hence, for $n \geq 0$, the following identity is true.

$$\mathcal{H}W_n = 5\mathcal{H}W_{n-1} - 8\mathcal{H}W_{n-2} + 4\mathcal{H}W_{n-3}. \tag{2.2}$$

The sequence $\{\mathcal{H}W_n\}_{n\geq 0}$ can be extended to negative subscripts by defining

$$\mathcal{H}W_{-n} = -2\mathcal{H}W_{-(n-1)} - \frac{5}{4}\mathcal{H}W_{-(n-2)} + \frac{1}{4}\mathcal{H}W_{-(n-3)}.$$

for $n = 1, 2, 3, ...$ respectively. Therefore, recurrence (2.2) holds for all integer n.

The few hyperbolic generalized Woodall numbers with positive subscript and negative subscript are given in the following Table 2.

Table 2. A few hyperbolic generalized Woodall numbers

n	$\mathcal{H}W_n$	$\mathcal{H}W_{-n}$
0	$\mathcal{H}W_0$	$\mathcal{H}W_0$
1	$\mathcal{H}W_1$	$\frac{1}{4}(8\mathcal{H}W_0 - 5\mathcal{H}W_1 + \mathcal{H}W_2)$
2	$\mathcal{H}W_2$	$\frac{1}{4}(11\mathcal{H}W_0 - 9\mathcal{H}W_1 + 2\mathcal{H}W_2)$
3	$4\mathcal{H}W_0 - 8\mathcal{H}W_1 + 5\mathcal{H}W_2$	$\frac{1}{16}(52\mathcal{H}W_0 - 47\mathcal{H}W_1 + 11\mathcal{H}W_2)$
4	$20\mathcal{H}W_0 - 36\mathcal{H}W_1 + 17\mathcal{H}W_2$	$\frac{1}{16}(57\mathcal{H}W_0 - 54\mathcal{H}W_1 + 13\mathcal{H}W_2)$
5	$68\mathcal{H}W_0 - 116\mathcal{H}W_1 + 49\mathcal{H}W_2$	$\frac{1}{64}(240\mathcal{H}W_0 - 233\mathcal{H}W_1 + 57\mathcal{H}W_2)$

Note that

$$\begin{aligned}
\mathcal{H}W_0 &= W_0 + jW_1, \\
\mathcal{H}W_1 &= W_1 + jW_2, \\
\mathcal{H}W_2 &= W_2 + jW_3 = W_2 + j(4W_0 - 8W_1 + 5W_2).
\end{aligned}$$

For hyperbolic modified Woodall numbers (taking $W_n = G_n$, $G_0 = 0$, $G_1 = 1$, $G_2 = 5$), we get

$$\begin{aligned}
\mathcal{H}G_0 &= G_0 + jG_1 = j, \\
\mathcal{H}G_1 &= G_1 + jG_2 = 1 + 5j, \\
\mathcal{H}G_2 &= G_2 + jG_3 = 5 + 17j
\end{aligned}$$

and for hyperbolic modified Cullen numbers (taking $W_n = H_n$, $H_0 = 3$, $H_1 = 5$, $H_2 = 9$), we get

$$\begin{aligned}
\mathcal{H}H_0 &= H_0 + jH_1 = 3 + 5j, \\
\mathcal{H}H_1 &= H_1 + jH_2 = 5 + 9j, \\
\mathcal{H}H_2 &= H_2 + jH_3 = 9 + 17j
\end{aligned}$$

and for hyperbolic Woodall numbers (taking $W_n = R_n$, $R_0 = -1$, $R_1 = 1$, $R_2 = 7$), we get

$$\begin{aligned}\mathcal{H}R_0 &= R_0 + jR_1 = -1 + j,\\ \mathcal{H}R_1 &= R_1 + jR_2 = 1 + 7j,\\ \mathcal{H}R_2 &= R_2 + jR_3 = 7 + 23j\end{aligned}$$

and for hyperbolic Cullen numbers (taking $W_n = C_n$, $C_0 = 1$, $C_1 = 3$, $C_2 = 9$), we get

$$\begin{aligned}\mathcal{H}C_0 &= C_0 + jC_1 = 1 + 3j,\\ \mathcal{H}C_1 &= C_1 + jC_2 = 3 + 9j,\\ \mathcal{H}C_2 &= C_2 + jC_3 = 9 + 25j.\end{aligned}$$

A few hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers with positive subscript and negative subscript are given in Table 3, Table 4, Table 5 and Table 6.

Table 3. Hyperbolic modified Woodall numbers

n	$\mathcal{H}G_n$	$\mathcal{H}G_{-n}$
0	j	j
1	$1+5j$	0
2	$5+17j$	$\frac{1}{4}$
3	$17+49j$	$\frac{1}{2}+\frac{1}{4}j$
4	$49+129j$	$\frac{11}{16}+\frac{1}{2}j$
5	$129+321j$	$\frac{13}{16}+\frac{11}{16}j$

Table 4. Hyperbolic modified Cullen numbers

n	$\mathcal{H}H_n$	$\mathcal{H}H_{-n}$
0	$3+5j$	$3+5j$
1	$5+9j$	$2+3j$
2	$9+17j$	$\frac{3}{2}+3j$
3	$17+33j$	$\frac{5}{4}+\frac{3}{2}j$
4	$33+65j$	$\frac{9}{8}+\frac{3}{2}j$
5	$65+129j$	$\frac{17}{16}+\frac{9}{8}j$

Table 5. Hyperbolic Woodall numbers

n	$\mathcal{H}R_n$	$\mathcal{H}R_{-n}$
0	$-1+j$	$-1+j$
1	$1+7j$	$-\frac{3}{2}-j$
2	$7+23j$	$-\frac{3}{2}-\frac{3}{2}j$
3	$23+63j$	$-\frac{11}{8}-\frac{3}{2}j$
4	$63+159j$	$-\frac{5}{4}-\frac{11}{8}j$
5	$159+383j$	$-\frac{37}{32}-\frac{5}{4}j$

Table 6. Hyperbolic Cullen numbers

n	$\mathcal{H}C_n$	$\mathcal{H}C_{-n}$
0	$1+3j$	$1+3j$
1	$3+9j$	$\frac{1}{2}+j$
2	$9+25j$	$\frac{1}{2}+j$
3	$25+65j$	$\frac{5}{8}+\frac{1}{2}j$
4	$65+161j$	$\frac{3}{4}+\frac{1}{2}j$
5	$161+385j$	$\frac{27}{32}+\frac{3}{4}j$

Now, we will state Binet's formula for the hyperbolic generalized Woodall numbers and in the rest of the paper, we fix the following notations:

$$\begin{aligned}\widehat{\alpha} &= 1+2j,\\ \widehat{\beta} &= 2j,\\ \widehat{\gamma} &= 1+j.\end{aligned}$$

Note that we have the following identities:

$$\begin{aligned}\widehat{\alpha}^2 &= 5+4j,\\ \widehat{\beta}^2 &= 4,\\ \widehat{\gamma}^2 &= 2+2j,\\ \widehat{\alpha}\widehat{\beta} &= 4+2j,\\ \widehat{\alpha}\widehat{\gamma} &= 3+3j,\\ \widehat{\beta}\widehat{\gamma} &= 2+2j,\\ \widehat{\alpha}\widehat{\beta}\widehat{\gamma} &= 6+6j.\end{aligned}$$

2.1 Binet's Formula

Now, we present Binet's formula in the following theorem.

Theorem 2.1. *(Binet's Formula) For any integer n, the nth hyperbolic generalized Woodall number is*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.3}$$

Proof. Using Binet's formula given below

$$W_n = (A_1 + A_2 n)2^n + A_3,$$

we obtain

$$\begin{aligned}\mathcal{H}W_n &= W_n + jW_{n+1}\\ &= (A_1 + A_2 n)2^n + A_3 + j((A_1 + A_2(n+1))2^{n+1} + A_3)\\ &= A_1 2^n + A_2 n 2^n + A_3\\ &\quad + jA_1 2^{n+1} + jA_2 n 2^{n+1} + jA_2 2^{n+1} + jA_3\\ &= A_1 2^n(1+2j) + A_2 n 2^n(1+2j) + A_2 2^n(2j) + A_3(1+j)\\ &= A_1 2^n\widehat{\alpha} + A_2 n 2^n\widehat{\alpha} + A_2 2^n\widehat{\beta} + A_3\widehat{\gamma}\\ &= (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}.\end{aligned}$$

This proves (2.3). □

As special cases, for any integer n, the Binet's Formula of nth hyperbolic modified Woodall number, hyperbolic modified Cullen number, hyperbolic Woodall number and hyperbolic Cullen number are

- $\mathcal{H}G_n = (-\widehat{\alpha} + \widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathcal{H}G_n = 1 + (n-1)2^n + j(1 + n2^{n+1})$.
- $\mathcal{H}H_n = (2\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathcal{H}H_n = 1 + 2^{n+1} + j(1 + 2^{n+2})$.
- $\mathcal{H}R_n = (\widehat{\beta} + n\widehat{\alpha})2^n - \widehat{\gamma}$,
  $\mathcal{H}R_n = -1 + n2^n + j(-1 + 2^{n+1} + n2^{n+1})$.
- $\mathcal{H}C_n = (\widehat{\beta} + n\widehat{\alpha})2^n + \widehat{\gamma}$,
  $\mathcal{H}C_n = 1 + n2^n + j(1 + 2^{n+1} + n2^{n+1})$.

Next, we present generating function of the hyperbolic generalized Woodall numbers.

2.2 Generating Function

Theorem 2.2. *The generating function for the hyperbolic generalized Woodall numbers is*

$$\sum_{n=0}^{\infty}\mathcal{H}W_n x^n = \frac{\mathcal{H}W_0+(\mathcal{H}W_1-5\mathcal{H}W_0)x+(\mathcal{H}W_2-5\mathcal{H}W_1+8\mathcal{H}W_0)x^2}{1-5x+8x^2-4x^3}. \tag{2.4}$$

Proof. Let

$$g(x)=\sum_{n=0}^{\infty}\mathcal{H}W_n x^n$$

be generating function of the hyperbolic generalized Woodall numbers. Then, using the definition of the hyperbolic generalized Woodall numbers, and substracting $xg(x)$, $x^2g(x)$ and $x^3g(x)$ from $g(x)$, we obtain (note the shift in the index n in the third line)

$$\begin{aligned}
(1-5x+8x^2-4x^3)g(x) &= \sum_{n=0}^{\infty}\mathcal{H}W_n x^n-5x\sum_{n=0}^{\infty}\mathcal{H}W_n x^n+8x^2\sum_{n=0}^{\infty}\mathcal{H}W_n x^n-4x^3\sum_{n=0}^{\infty}\mathcal{H}W_n x^n\\
&= \sum_{n=0}^{\infty}\mathcal{H}W_n x^n-5\sum_{n=0}^{\infty}\mathcal{H}W_n x^{n+1}+8\sum_{n=0}^{\infty}\mathcal{H}W_n x^{n+2}-4\sum_{n=0}^{\infty}\mathcal{H}W_n x^{n+3}\\
&= \sum_{n=0}^{\infty}\mathcal{H}W_n x^n-5\sum_{n=1}^{\infty}\mathcal{H}W_{n-1} x^{n}+8\sum_{n=2}^{\infty}\mathcal{H}W_{n-2} x^{n}-4\sum_{n=3}^{\infty}\mathcal{H}W_{n-3} x^{n}\\
&= (\mathcal{H}W_0+\mathcal{H}W_1x+\mathcal{H}W_2x^2)-5(\mathcal{H}W_0x+\mathcal{H}W_1x^2)+8\mathcal{H}W_0x^2\\
&\quad+\sum_{n=3}^{\infty}(\mathcal{H}W_n-5\mathcal{H}W_{n-1}+8\mathcal{H}W_{n-2}-4\mathcal{H}W_{n-3})x^n\\
&= \mathcal{H}W_0+(\mathcal{H}W_1-5\mathcal{H}W_0)x+(\mathcal{H}W_2-5\mathcal{H}W_1+8\mathcal{H}W_0)x^2.
\end{aligned}$$

Note that, we use the recurrence relation $\mathcal{H}W_n=5\mathcal{H}W_{n-1}-8\mathcal{H}W_{n-2}+4\mathcal{H}W_{n-3}$. Rearranging above equation, we get

$$g(x)=\frac{\mathcal{H}W_0+(\mathcal{H}W_1-5\mathcal{H}W_0)x+(\mathcal{H}W_2-5\mathcal{H}W_1+8\mathcal{H}W_0)x^2}{1-5x+8x^2-4x^3}.$$

The proof is finished. □

As special cases, the generating functions for the hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers are

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{H}G_n x^n &= \frac{j+x}{1-5x+8x^2-4x^3},\\
\sum_{n=0}^{\infty}\mathcal{H}H_n x^n &= \frac{5j+3+(-16j-10)\,x+(12j+8)\,x^2}{1-5x+8x^2-4x^3},\\
\sum_{n=0}^{\infty}\mathcal{H}R_n x^n &= \frac{-1+j+(2j+6)x+(-4j-6)x^2}{1-5x+8x^2-4x^3}
\end{aligned}$$

and

$$\sum_{n=0}^{\infty}\mathcal{H}C_n x^n=\frac{3j+1+(-6j-2)\,x+(4j+2)\,x^2}{1-5x+8x^2-4x^3}$$

respectively.

Now, we obtained the Binet formula using the generating function.

2.3 Obtaining Binet's Formula From Generating Function

We obtain the Binet's formula of hyperbolic generalized Woodall number $\{\mathcal{H}W_n\}$ by the use of generating function for $\mathcal{H}W_n$.

Theorem 2.3. *(The Binet's formula of hyperbolic generalized Woodall numbers)*

$$\mathcal{H}W_n = (A_1\widehat{\alpha} + A_2\widehat{\beta} + A_2 n\widehat{\alpha})2^n + A_3\widehat{\gamma}. \tag{2.5}$$

Proof. Let

$$\sum_{n=0}^{\infty} \mathcal{H}W_n x^n = \frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{1 - 5x + 8x^2 - 4x^3}.$$

Then, we write

$$\frac{\mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2}{(1-x)(1-2x)^2} = \frac{d_1}{(1-x)} + \frac{d_2}{(1-2x)} + \frac{d_3}{(1-2x)^2}. \tag{2.6}$$

So

$$\begin{aligned} \mathcal{H}W_0 + (\mathcal{H}W_1 - 5\mathcal{H}W_0)x + (\mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0)x^2 &= (d_1 + d_2 + d_3) + (-4d_1 - 3d_2 - d_3)x \\ &\quad +(4d_1 + 2d_2)x^2. \end{aligned}$$

We get

$$\begin{aligned} \mathcal{H}W_0 &= d_1 + d_2 + d_3, \\ \mathcal{H}W_1 - 5\mathcal{H}W_0 &= -4d_1 - 3d_2 - d_3, \\ \mathcal{H}W_2 - 5\mathcal{H}W_1 + 8\mathcal{H}W_0 &= 4d_1 + 2d_2. \end{aligned}$$

If we solve these simultanious equation,

$$\begin{aligned} d_1 &= 4\mathcal{H}W_0 - 4\mathcal{H}W_1 + \mathcal{H}W_2, \\ d_2 &= -4\mathcal{H}W_0 + \frac{11}{2}\mathcal{H}W_1 - \frac{3}{2}\mathcal{H}W_2, \\ d_3 &= \mathcal{H}W_0 - \frac{3}{2}\mathcal{H}W_1 + \frac{1}{2}\mathcal{H}W_2. \end{aligned}$$

Thus (2.6) can be written as

$$\begin{aligned}
\sum_{n=0}^{\infty}\mathcal{H}W_n x^n &= d_1\frac{1}{(1-x)}+d_2\frac{1}{(1-2x)}+d_3\frac{1}{(2x-1)^2},\\
&= d_1\sum_{n=0}^{\infty}x^n+d_2\sum_{n=0}^{\infty}2^n x^n+d_3\sum_{n=0}^{\infty}2^n(n+1)x^n,\\
&= \sum_{n=0}^{\infty}(d_1+d_2 2^n+d_3 2^n(n+1))x^n,\\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n\\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n(n+1))x^n,\\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(-4\mathcal{H}W_0+\frac{11}{2}\mathcal{H}W_1-\frac{3}{2}\mathcal{H}W_2)2^n\\
&\quad +(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)2^n n)x^n,\\
&= \sum_{n=0}^{\infty}(4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n2^n\\
&\quad +(-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)2^n)x^n,\\
&= \sum_{n=0}^{\infty}((-3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2)+(\mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2)n)2^n\\
&\quad +4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2)x^n.
\end{aligned}$$

This gives

$$\mathcal{H}W_n=(\mathcal{H}A_1+\mathcal{H}A_2 n)2^n+\mathcal{H}A_3$$

where

$$\begin{aligned}
\mathcal{H}A_1 &= -3\mathcal{H}W_0+4\mathcal{H}W_1-\mathcal{H}W_2,\\
\mathcal{H}A_2 &= \mathcal{H}W_0-\frac{3}{2}\mathcal{H}W_1+\frac{1}{2}\mathcal{H}W_2,\\
\mathcal{H}A_3 &= 4\mathcal{H}W_0-4\mathcal{H}W_1+\mathcal{H}W_2.
\end{aligned}$$

Note that the following equalities are true:

$$\begin{aligned}
A_1\widehat{\alpha}+A_2\widehat{\beta} &= (-W_2+4W_1-3W_0)(1+2j)+(\frac{W_2-3W_1+2W_0}{2})(2j)\\
&= -3W_0+4W_1-W_2+j(-4W_0+5W_1-W_2).
\end{aligned}$$

$$\begin{aligned}
A_2\widehat{\alpha} &= \frac{W_2-3W_1+2W_0}{2}(1+2j)\\
&= W_0-\frac{3}{2}W_1+\frac{1}{2}W_2+j(2W_0-3W_1+W_2).
\end{aligned}$$

$$A_3\widehat{\gamma}=W_2-4W_1+4W_0+j(W_2-4W_1+4W_0).$$

Therefore, we can write the following equalition:

$$\mathcal{H}W_n=(A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2 n\widehat{\alpha})2^n+A_3\widehat{\gamma}.$$

The proof is finished. □

Next, using Theorem 2.3, we present the Binet's formulas of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers.

3 SOME IDENTITIES FOR HYPERBOLIC GENERALIZED WOODALL NUMBERS

We now present a few special identities for the hyperbolic generalized Woodall sequence $\{\mathcal{H}W_n\}$. The following theorem presents the Simpson's identity for the hyperbolic generalized Woodall numbers.

Theorem 3.1. *(Simpson's formula for hyperbolic generalized Woodall sequence)For all integers n we have*

$$\begin{vmatrix} \mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n \\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1} \\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2} \end{vmatrix} = 4^n \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.$$

Proof. For the proof we use mathematical induction. For $n = 0$ identity is true. First, we proof the identity for $n \geq 0$. Now, we obtain the identity is true for $n = k$. Hence, we write the following identity

$$\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \\ \mathcal{H}W_k & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^k \begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.$$

For $n = k + 1$, we get

$$\begin{aligned}
\begin{vmatrix} \mathcal{H}W_{k+3} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} &= \begin{vmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_k & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ 5\mathcal{H}W_{k+1} - 8\mathcal{H}W_k + 4\mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ 5\mathcal{H}W_k - 8\mathcal{H}W_{k-1} + 4\mathcal{H}W_{k-2} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&= 5\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k+1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_k & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} - 8\begin{vmatrix} \mathcal{H}W_{k+1} & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_k & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k-1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&\quad +4\begin{vmatrix} \mathcal{H}W_k & \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} \\ \mathcal{H}W_{k-1} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k-2} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \end{vmatrix} \\
&= 4\begin{vmatrix} \mathcal{H}W_{k+2} & \mathcal{H}W_{k+1} & \mathcal{H}W_k \\ \mathcal{H}W_{k+1} & \mathcal{H}W_k & \mathcal{H}W_{k-1} \\ \mathcal{H}W_k & \mathcal{H}W_{k-1} & \mathcal{H}W_{k-2} \end{vmatrix} = 4^{k+1}\begin{vmatrix} \mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0 \\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1} \\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2} \end{vmatrix}.
\end{aligned}$$

The other case can be done similarly. Thus, the proof is finished. □

From prewious theorem, we get following corollary.

Corollary 3.2. *(Simpson's formula for hyperbolic generalized Woodall sequence's special cases)*

(a) $\begin{vmatrix} \mathcal{H}G_{k+2} & \mathcal{H}G_{k+1} & \mathcal{H}G_k \\ \mathcal{H}G_{k+1} & \mathcal{H}G_k & \mathcal{H}G_{k-1} \\ \mathcal{H}G_k & \mathcal{H}G_{k-1} & \mathcal{H}G_{k-2} \end{vmatrix} = -4^{n-1}(9 + 9j).$

(b) $\begin{vmatrix} \mathcal{H}H_{k+2} & \mathcal{H}H_{k+1} & \mathcal{H}H_k \\ \mathcal{H}H_{k+1} & \mathcal{H}H_k & \mathcal{H}H_{k-1} \\ \mathcal{H}H_k & \mathcal{H}H_{k-1} & \mathcal{H}H_{k-2} \end{vmatrix} = 0.$

(c) $\begin{vmatrix} \mathcal{H}R_{k+2} & \mathcal{H}R_{k+1} & \mathcal{H}R_k \\ \mathcal{H}R_{k+1} & \mathcal{H}R_k & \mathcal{H}R_{k-1} \\ \mathcal{H}R_k & \mathcal{H}R_{k-1} & \mathcal{H}R_{k-2} \end{vmatrix} = 4^{n-1}(9 + 9j).$

(d) $\begin{vmatrix} \mathcal{H}C_{k+2} & \mathcal{H}C_{k+1} & \mathcal{H}C_k \\ \mathcal{H}C_{k+1} & \mathcal{H}C_k & \mathcal{H}C_{k-1} \\ \mathcal{H}C_k & \mathcal{H}C_{k-1} & \mathcal{H}C_{k-2} \end{vmatrix} = -4^{n-1}(9 + 9j).$

Theorem 3.3. *(Catalan's identity) For all integers n and m, the following identity holds*
$\mathcal{H}W_{n+m}\mathcal{H}W_{n-m}-\mathcal{H}W_n^2 = 2^{n-m}(-2^{m+n}m^2\widehat{\alpha}^2A_2^2+A_2A_3(-2^{m+1}\widehat{\beta}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}+2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma})+A_1A_3(\widehat{\alpha}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma}))$.

Proof. Using the Binet's formula $\mathcal{H}W_n = (A_1\widehat{\alpha}+A_2\widehat{\beta}+A_2n\widehat{\alpha})2^n+A_3\widehat{\gamma}$, we get the required identity. □

As special cases of the above theorem, we give Catalan's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Catalan's identity of hyperbolic Woodall numbers.

Corollary 3.4. *(Catalan's identity for the hyperbolic modified Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}G_{n+m}\mathcal{H}G_{n-m}-\mathcal{H}G_n^2 &= -2^{n-m}(\widehat{\alpha}\widehat{\gamma}-\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\alpha}\widehat{\gamma}-2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\alpha}\widehat{\gamma}+2^{m+1}\widehat{\beta}\widehat{\gamma}\\ &\quad +m\widehat{\alpha}\widehat{\gamma}-n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2-2^{2m}m\widehat{\alpha}\widehat{\gamma}-2^{2m}n\widehat{\alpha}\widehat{\gamma}+2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = G_n$ in Theorem 3.3. □

Secondly, we give Catalan's identity of hyperbolic modified Cullen numbers.

Corollary 3.5. *(Catalan's identity for the hyperbolic modified Cullen numbers) For all integers n and m, the following identity holds*

$$\mathcal{H}H_{n+m}\mathcal{H}H_{n-m}-\mathcal{H}H_n^2 = 2^{n-m}(2\widehat{\alpha}\widehat{\gamma}+2\times 2^{2m}\widehat{\alpha}\widehat{\gamma}-2\times 2^{m+1}\widehat{\alpha}\widehat{\gamma}).$$

Proof. Take $W_n = H_n$ in Theorem 3.3. □

Thirdly, we give Catalan's identity of hyperbolic Woodall numbers.

Corollary 3.6. *(Catalan's identity for the hyperbolic Woodall numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}R_{n+m}\mathcal{H}R_{n-m}-\mathcal{H}R_n^2 &= -2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}+2^{m+n}m^2\widehat{\alpha}^2\\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = R_n$ in Theorem 3.3. □

Fourthly, we give Catalan's identity of hyperbolic Cullen numbers.

Corollary 3.7. *(Catalan's identity for the hyperbolic Cullen numbers) For all integers n and m, the following identity holds*

$$\begin{aligned}\mathcal{H}C_{n+m}\mathcal{H}C_{n-m}-\mathcal{H}C_n^2 &= 2^{n-m}(\widehat{\beta}\widehat{\gamma}+2^{2m}\widehat{\beta}\widehat{\gamma}-2^{m+1}\widehat{\beta}\widehat{\gamma}-m\widehat{\alpha}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma}-2^{m+n}m^2\widehat{\alpha}^2\\ &\quad +2^{2m}m\widehat{\alpha}\widehat{\gamma}+2^{2m}n\widehat{\alpha}\widehat{\gamma}-2^{m+1}n\widehat{\alpha}\widehat{\gamma}).\end{aligned}$$

Proof. Take $W_n = C_n$ in Theorem 3.3. □

Note that for $m = 1$ in Catalan's identity, we get the Cassini's identity for the hyperbolic generalized Woodall sequence.

Corollary 3.8. *(Cassini's identity) For all integers n, the following identity holds*

$$\mathcal{H}W_{n+1}\mathcal{H}W_{n-1}-\mathcal{H}W_n^2 = 2^{n-1}(A_2A_3(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+n\widehat{\alpha}\widehat{\gamma})-2^{n+1}A_2^2\widehat{\alpha}^2+A_1A_3\widehat{\alpha}\widehat{\gamma}).$$

As special cases of Cassini's identity, we give Cassini's identity of hyperbolic modified Woodall, hyperbolic modified Cullen, hyperbolic Woodall and hyperbolic Cullen numbers. Firstly, we present Cassini's identity of hyperbolic modified Woodall numbers.

Firstly, we give Cassini's identity of hyperbolic modified Woodall numbers.

Corollary 3.9. *(Cassini's identity of hyperbolic modified Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}G_{n+1}\mathcal{H}G_{n-1}-\mathcal{H}G_n^2=2^{n-1}(2\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Secondly, we give Cassini's identity of hyperbolic modified Cullen numbers.

Corollary 3.10. *(Cassini's identity of hyperbolic modified Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}H_{n+1}\mathcal{H}H_{n-1}-\mathcal{H}H_n^2=2^n\widehat{\alpha}\widehat{\gamma}.$$

Fourth, we give Cassini's identity of hyperbolic Woodall numbers.

Corollary 3.11. *(Cassini's identity of hyperbolic Woodall numbers) For all integers n, the following identity holds*

$$\mathcal{H}R_{n+1}\mathcal{H}R_{n-1}-\mathcal{H}R_n^2=-2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}+2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Third, we give Cassini's identity of hyperbolic Cullen numbers.

Corollary 3.12. *(Cassini's identity of hyperbolic Cullen numbers) For all integers n, the following identity holds*

$$\mathcal{H}C_{n+1}\mathcal{H}C_{n-1}-\mathcal{H}C_n^2=2^{n-1}(3\widehat{\alpha}\widehat{\gamma}+\widehat{\beta}\widehat{\gamma}-2^{n+1}\widehat{\alpha}^2+n\widehat{\alpha}\widehat{\gamma}).$$

Theorem 3.13. *For all integers m, n, G_n is woodall numbers, the following identity is true:*

$$\mathcal{H}W_{n+m}=\mathcal{H}W_nG_{m+1}+\mathcal{H}W_{n-1}(-8G_m+4G_{m-1})+4\mathcal{H}W_{n-2}G_m.$$

Proof. The identity (3.13) can be proved by mathematical induction on m. First of all, we assume that $m\geq 0$ and $n\geq 0$. If $m=0$ we get

$$\mathcal{H}W_n=\mathcal{H}W_nG_1+\mathcal{H}W_{n-1}(-8G_0+4G_{-1})+4\mathcal{H}W_{n-2}G_0$$

which is true by seeing that $G_{-1}=0, G_{-2}=\frac{1}{4}, G_{-3}=\frac{1}{2}$. We assume that the identity given holds for $m=k$. For $m=k+1$, we get

$$\begin{aligned}
\mathcal{H}W_{(k+1)+n} &= 5\mathcal{H}W_{n+k}-8\mathcal{H}W_{n+k-1}+4\mathcal{H}W_{n+k-2}\\
&= 5(\mathcal{H}W_nG_{k+1}+\mathcal{H}W_{n-1}(-8G_k+4G_{k-1})+4\mathcal{H}W_{n-2}G_k)\\
&\quad -8(\mathcal{H}W_nG_k+\mathcal{H}W_{n-1}(-8G_{k-1}+4G_{k-2})+4\mathcal{H}W_{n-2}G_{k-1})\\
&\quad +4(\mathcal{H}W_nG_{k-1}+\mathcal{H}W_{n-1}(-8G_{k-2}+4G_{k-3})+4\mathcal{H}W_{n-2}G_{k-2})\\
&= \mathcal{H}W_n(5G_{k+1}-8G_k+4G_{k-1})+\mathcal{H}W_{n-1}(-8(5G_k-8G_{k-1}+4G_{k-2})\\
&\quad +4(5G_{k-1}-8G_{k-2}+4G_{k-3}))+4\mathcal{H}W_{n-2}(5G_k-8G_{k-1}+4G_{k-2})\\
&= \mathcal{H}W_nG_{k+2}+\mathcal{H}W_{n-1}(-8G_{k+1}+4G_k)+4\mathcal{H}W_{n-2}G_{k+1}\\
&= \mathcal{H}W_nG_{(k+1)+1}+\mathcal{H}W_{n-1}(-8G_{(k+1)}+4G_{(k+1)-1})+4\mathcal{H}W_{n-2}G_{(k+1)}.
\end{aligned}$$

Consequently, by mathematical induction on m, this proves (3.13). Similarly, we can show for the other cases. So the proof is finished. □

4 LINEAR SUMS FOR HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we give the summation formulas of the hyperbolic generalized Woodall numbers with positive and negatif subscripts. Now, we present the summation formulas of the generalized Woodall numbers.

Proposition 4.1. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n}W_k=\frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9).$

- $\sum_{k=0}^{n} W_{k+1} = \frac{1}{2}W_2(2n + 2^{n+3}(n-1) - 2^{n+2}n + 8) - \frac{1}{2}W_1(8n - 2^{n+2}(3n-2) + 2^{n+3}(3n-5) + 30) + W_0(4n - 2^{n+2}(n-1) + 2^{n+3}(n-2) + 12).$
- $\sum_{k=0}^{n} W_{k+2} = \frac{1}{2}W_2(2n - 2^{n+3}(n+1) + 2^{n+4}n + 10) + W_0(4n + 2^{n+4}(n-1) - 2^{n+3}n + 16) - \frac{1}{2}W_1(8n - 2^{n+3}(3n+1) + 2^{n+4}(3n-2) + 40).$
- $\sum_{k=0}^{n} W_{k+3} = W_0(4n - 2^{n+4}(n+1) + 2^{n+5}n + 20) - \frac{1}{2}W_1(8n + 2^{n+5}(3n+1) - 2^{n+4}(3n+4) + 48) + \frac{1}{2}W_2(2n - 2^{n+4}(n+2) + 2^{n+5}(n+1) + 10).$

Proof. For the proof, see Soykan [52]. □

Proposition 4.2. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{2k} = \frac{1}{9}W_0(36n - 2^{2n+2}(2n-1) + 2^{2n+4}(2n-3) + 53) - \frac{1}{18}W_1(72n - 2^{2n+2}(6n-2) + 2^{2n+4}(6n-8) + 120) + \frac{1}{18}W_2(18n + 2^{2n+4}(2n-2) - 2 \times 2^{2n+2}n + 32).$
- $\sum_{k=0}^{n} W_{2k+1} = \frac{1}{18}W_2(18n - 2^{2n+3}(2n+1) + 2^{2n+5}(2n-1) + 40) - \frac{1}{18}W_1(72n - 2^{2n+3}(6n+1) + 2^{2n+5}(6n-5) + 150) + \frac{1}{9}W_0(36n + 2^{2n+5}(2n-2) - 2 \times 2^{2n+3}n + 64).$
- $\sum_{k=0}^{n} W_{2k+2} = \frac{1}{9}W_0(36n - 2^{2n+4}(2n+1) + 2^{2n+6}(2n-1) + 80) - \frac{1}{18}W_1(72n - 2^{2n+4}(6n+4) + 2^{2n+6}(6n-2) + 192) + \frac{1}{18}W_2(18n - 2^{2n+4}(2n+2) + 2 \times 2^{2n+6}n + 50).$
- $\sum_{k=0}^{n} W_{2k+3} = \frac{1}{18}W_2\left((18n - 2^{2n+5}(2n+3) + 2^{2n+7}(2n+1) + 58\right) - \frac{1}{18}W_1(72n + 2^{2n+7}(6n+1) - 2^{2n+5}(6n+7) + 240) + \frac{1}{9}W_0(36n - 2^{2n+5}(2n+2) + 2 \times 2^{2n+7}n + 100).$
- $\sum_{k=0}^{n} W_{2k+4} = \frac{1}{18}W_2(18n - 2^{2n+6}(2n+4) + 2^{2n+8}(2n+2) + 50) + \frac{1}{9}W_0(36n - 2^{2n+6}(2n+3) + 2^{2n+8}(2n+1) + 116) - \frac{1}{18}W_1(72n + 2^{2n+8}(6n+4) - 2^{2n+6}(6n+10) + 264).$

Proof. For the proof, see Soykan [52]. □

Proposition 4.3. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-k} = 4W_0(n + \frac{1}{2^{n+1}}(n+4) - \frac{1}{2^{n+2}}(n+3) - 1) + 2W_1(\frac{1}{2^{n+2}}(3n+8) - 2n - \frac{1}{2^{n+1}}(3n+11) + \frac{7}{2}) + 2W_2(\frac{1}{2}n + \frac{1}{2^{n+1}}(n+3) - \frac{1}{2^{n+2}}(n+2) - 1).$
- $\sum_{k=0}^{n} W_{-k+1} = 2W_2(\frac{1}{2}n + \frac{1}{2^{n}}(n+2) - \frac{1}{2^{n+1}}(n+1) - \frac{3}{2}) + 4W_0(n + \frac{1}{2^{n}}(n+3) - \frac{1}{2^{n+1}}(n+2) - 2) + 2W_1(\frac{1}{2^{n+1}}(3n+5) - 2n - \frac{1}{2^{n}}(3n+8) + 6).$
- $\sum_{k=0}^{n} W_{-k+2} = 2W_2(\frac{1}{2}n + 2^{1-n}(n+1) - \frac{1}{2^{n}}n - \frac{3}{2}) + 4W_0(n - \frac{1}{2^{n}}(n+1) + 2^{1-n}(n+2) - 3) - 2W_1(2n + 2^{1-n}(3n+5) - \frac{1}{2^{n}}(3n+2) - 8).$
- $\sum_{k=0}^{n} W_{-k+3} = 2W_2(\frac{1}{2}n + 2^{2-n}n - 2^{1-n}(n-1) + \frac{1}{2}) + 2W_1(2^{1-n}(3n-1) - 2n - 2^{2-n}(3n+2) + 6) + 4W_0(n - 2^{1-n}n + 2^{2-n}(n+1) - 3).$

Proof. For the proof, see Soykan [52]. □

Proposition 4.4. *For the generalized Woodall numbers, we have the following formulas:*

- $\sum_{k=0}^{n} W_{-2k} = \frac{8}{9}W_1(\frac{1}{2^{2n+4}}(6n+8) - \frac{9}{2}n - \frac{1}{2^{2n+2}}(6n+14) + 3) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+2}}(2n+5) - \frac{1}{2^{2n+4}}(2n+3) - \frac{1}{2}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+2}}(2n+4) - \frac{1}{2^{2n+4}}(2n+2) - \frac{7}{8}).$
- $\sum_{k=0}^{n} W_{-2k+1} = \frac{8}{9}W_1(\frac{1}{2^{2n+3}}(6n+5) - \frac{9}{2}n - \frac{1}{2^{2n+1}}(6n+11) + 6) + \frac{16}{9}W_0(\frac{9}{4}n + \frac{1}{2^{2n+1}}(2n+4) - \frac{1}{2^{2n+3}}(2n+2) - \frac{7}{4}) + \frac{8}{9}W_2(\frac{9}{8}n + \frac{1}{2^{2n+1}}(2n+3) - \frac{1}{2^{2n+3}}(2n+1) - \frac{11}{8}).$
- $\sum_{k=0}^{n} W_{-2k+2} = \frac{8}{9}W_2(\frac{9}{8}n - \frac{2}{2^{2n+2}}n + \frac{1}{2^{2n}}(2n+2) - \frac{7}{8}) - \frac{16}{9}W_0(\frac{1}{2^{2n+2}}(2n+1) - \frac{9}{4}n - \frac{1}{2^{2n}}(2n+3) + \frac{11}{4}) + \frac{8}{9}W_1(\frac{1}{2^{2n+2}}(6n+2) - \frac{9}{2}n - \frac{1}{2^{2n}}(6n+8) + \frac{15}{2}).$
- $\sum_{k=0}^{n} W_{-2k+3} = \frac{8}{9}W_1(\frac{1}{2^{2n+1}}(6n-1) - \frac{9}{2}n - 2^{1-2n}(6n+5) + \frac{3}{2}) + \frac{8}{9}W_2(\frac{9}{8}n - \frac{1}{2^{2n+1}}(2n-1) + 2^{1-2n}(2n+1) + \frac{25}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{1-2n}(2n+2) - \frac{2}{2^{2n+1}}n - \frac{7}{4}).$
- $\sum_{k=0}^{n} W_{-2k+4} = \frac{8}{9}W_2(\frac{9}{8}n + 2 \times 2^{2-2n}n - \frac{1}{2^{2n}}(2n-2) + \frac{137}{8}) + \frac{16}{9}W_0(\frac{9}{4}n + 2^{2-2n}(2n+1) - \frac{1}{2^{2n}}(2n-1) + \frac{25}{4}) - \frac{8}{9}W_1(\frac{9}{2}n + 2^{2-2n}(6n+2) - \frac{1}{2^{2n}}(6n-4) + \frac{57}{2}).$

Proof. For the proof, see Soykan [52]. □

Now, we present the formulas which give the summation of the hyperbolic generalized Woodall numbers.

Theorem 4.5. *For $n \geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n} \mathcal{H}W_k = (3+n-3\times 2^n+2^n n+4j+jn-2^{n+2}j+2^{n+1}jn)W_2+(-11-4n+11\times 2^n-3\times 2^n n-15j-4jn+2^{n+4}j-3\times 2^{n+1}jn)W_1+(9+4n-2^{n+3}+2^{n+1}n+12j+4jn-3\times 2^{n+2}j+2^{n+2}jn)W_0.$

(b) $\sum_{k=0}^{n} \mathcal{H}W_{2k} = (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}j+jn-\frac{5}{9}2^{2n+2}j+\frac{1}{3}2^{2n+3}jn)W_2+(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}j+\frac{7}{3}2^{2n+2}j-4jn-2^{2n+3}jn)W_1+(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}j-\frac{1}{9}2^{2n+6}j+4jn+\frac{1}{3}2^{2n+4}jn)W_0.$

(c) $\sum_{k=0}^{n} \mathcal{H}W_{2k+1} = (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}j-\frac{1}{9}2^{2n+4}j+jn+\frac{1}{3}2^{2n+4}jn)W_2+(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}j-\frac{32}{3}j-4jn-2^{2n+4}jn)W_1+(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}j-\frac{5}{9}2^{2n+4}j+4jn+\frac{1}{3}2^{2n+5}jn)W_0.$

Proof. Proof can be obtained by using Proposition 4.4.

(a) We can derive the following using the formulas in Proposition 4.1.

$$\sum_{k=0}^{n} \mathcal{H}W_k = \sum_{k=0}^{n} W_k + j\sum_{k=0}^{n} W_{k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_k &\\
={}& \frac{1}{2}W_2(2n-2^{n+1}(n-1)+2^{n+2}(n-2)+6)-\frac{1}{2}W_1(8n-2^{n+1}(3n-5)\\
&+2^{n+2}(3n-8)+22)+W_0(4n-2^{n+1}(n-2)+2^{n+2}(n-3)+9)\\
&+j(\frac{1}{2}W_2(2n+2^{n+3}(n-1)-2^{n+2}n+8)-\frac{1}{2}W_1(8n-2^{n+2}(3n-2)\\
&+2^{n+3}(3n-5)+30)+W_0(4n-2^{n+2}(n-1)+2^{n+3}(n-2)+12)).
\end{aligned}$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_k &\\
={}& (3+n-3\times 2^n+2^n n+4j+jn-2^{n+2}j+2^{n+1}jn)W_2\\
&+(-11-4n+11\times 2^n-3\times 2^n n-15j-4jn+2^{n+4}j-3\times 2^{n+1}jn)W_1\\
&+(9+4n-2^{n+3}+2^{n+1}n+12j+4jn-3\times 2^{n+2}j+2^{n+2}jn)W_0.
\end{aligned}$$

The proof is finished. □

(b) We can derive the following using the formulas in Proposition 4.2.

$$\sum_{k=0}^{n} \mathcal{H}W_{2k} = \sum_{k=0}^{n} W_{2k} + j\sum_{k=0}^{n} W_{2k+1}.$$

$$\begin{aligned}
\sum_{k=0}^{n} \mathcal{H}W_{2k} &\\
={}& \frac{1}{9}W_0(36n-2^{2n+2}(2n-1)+2^{2n+4}(2n-3)+53)-\frac{1}{18}W_1(72n-2^{2n+2}(6n-2)\\
&+2^{2n+4}(6n-8)+120)+\frac{1}{18}W_2(18n+2^{2n+4}(2n-2)-2\times 2^{2n+2}n+32)\\
&+j(\frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)\\
&+2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k}\\
&= (\frac{16}{9}+n-\frac{1}{9}2^{2n+4}+\frac{1}{3}2^{2n+2}n+\frac{20}{9}j+jn-\frac{5}{9}2^{2n+2}j+\frac{1}{3}2^{2n+3}jn)W_2\\
&\quad +(-\frac{20}{3}-4n+\frac{5}{3}2^{2n+2}-2^{2n+2}n-\frac{25}{3}j+\frac{7}{3}2^{2n+2}j-4jn-2^{2n+3}jn)W_1\\
&\quad +(\frac{53}{9}-\frac{11}{9}2^{2n+2}+4n+\frac{1}{3}2^{2n+3}n+\frac{64}{9}j-\frac{1}{9}2^{2n+6}j+4jn+\frac{1}{3}2^{2n+4}jn)W_0.
\end{aligned}$$

The proof is completed. □

(c) We can derive the following using the formulas in Proposition 4.4.

$$\sum_{k=0}^{n}\mathcal{H}W_{2k+1}=\sum_{k=0}^{n}W_{2k+1}+j\sum_{k=0}^{n}W_{2k+2}.$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k+1}\\
&= \frac{1}{18}W_2(18n-2^{2n+3}(2n+1)+2^{2n+5}(2n-1)+40)-\frac{1}{18}W_1(72n-2^{2n+3}(6n+1)\\
&\quad +2^{2n+5}(6n-5)+150)+\frac{1}{9}W_0(36n+2^{2n+5}(2n-2)-2\times 2^{2n+3}n+64)\\
&\quad +j(\frac{1}{9}W_0(36n-2^{2n+4}(2n+1)+2^{2n+6}(2n-1)+80)-\frac{1}{18}W_1(72n-2^{2n+4}(6n+4)\\
&\quad +2^{2n+6}(6n-2)+192)+\frac{1}{18}W_2(18n-2^{2n+4}(2n+2)+2\times 2^{2n+6}n+50)).
\end{aligned}$$

$$\begin{aligned}
&\sum_{k=0}^{n}\mathcal{H}W_{2k+1}\\
&= (\frac{20}{9}-\frac{5}{9}2^{2n+2}+n+\frac{1}{3}2^{2n+3}n+\frac{25}{9}j-\frac{1}{9}2^{2n+4}j+jn+\frac{1}{3}2^{2n+4}jn)W_2\\
&\quad +(-\frac{25}{3}+\frac{7}{3}2^{2n+2}-4n-2^{2n+3}n+\frac{1}{3}2^{2n+5}j-\frac{32}{3}j-4jn-2^{2n+4}jn)W_1\\
&\quad +(\frac{64}{9}-\frac{1}{9}2^{2n+6}+4n+\frac{1}{3}2^{2n+4}n+\frac{80}{9}j-\frac{5}{9}2^{2n+4}j+4jn+\frac{1}{3}2^{2n+5}jn)W_0.
\end{aligned}$$

The proof is finished. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.6. *For $n\geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}G_k=4+n+2^{n+1}n-2^{n+2}+j(5-5\times 2^{n+2}+n+2^{n+4}+2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}G_{2k}=\frac{20}{9}+n+\frac{2}{3}2^{2n+2}n+\frac{5}{3}2^{2n+2}-\frac{5}{9}2^{2n+4}+j(\frac{25}{9}-\frac{4}{9}2^{2n+2}+n+\frac{2}{3}2^{2n+3}n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}G_{2k+1}=\frac{25}{9}+n+\frac{2}{3}2^{2n+3}n-\frac{4}{9}2^{2n+2}+j(\frac{29}{9}-\frac{5}{9}2^{2n+4}+\frac{1}{3}2^{2n+5}+n+\frac{2}{3}2^{2n+4}n)$.

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

Corollary 4.7. *For $n\geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}H_k=-1+n-6\times 2^n n-3\times 2^{n+3}+3\times 2^{n+1}n+28\times 2^n+j(-3-18\times 2^{n+2}+5\times 2^{n+4}+n-6\times 2^{n+1}n+3\times 2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}H_{2k}=\frac{1}{3}+n-2^{2n+3}n+2^{2n+3}n+\frac{14}{3}2^{2n+2}-2^{2n+4}+j(-\frac{1}{3}+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}+n-2^{2n+4}n+2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}H_{2k+1}=-\frac{1}{3}+n-2\times 2^{2n+3}n+2^{2n+4}n+\frac{20}{3}2^{2n+2}-\frac{1}{3}2^{2n+6}+j(-\frac{5}{3}-\frac{8}{3}2^{2n+4}+\frac{5}{3}2^{2n+5}+n-2^{2n+5}n+2^{2n+5}n)$.

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.8. *For $n\geq 0$, hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}R_{k}=1-n+4\times 2^{n}n+2^{n+3}-2^{n+1}n-10\times 2^{n}+j(1-2^{n+4}+2^{n+4}-n+2^{n+3}n-2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}R_{2k}=-\frac{1}{9}-n+\frac{4}{3}2^{2n+2}n-\frac{1}{3}2^{2n+3}n+\frac{26}{9}2^{2n+2}-\frac{7}{9}2^{2n+4}+j(\frac{1}{9}-n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}R_{2k+1}=\frac{1}{9}-n+\frac{4}{3}2^{2n+3}n-\frac{1}{3}2^{2n+4}n-\frac{14}{9}2^{2n+2}+\frac{1}{9}2^{2n+6}+j(-\frac{1}{9}+\frac{1}{3}2^{2n+5}-\frac{2}{9}2^{2n+4}-n+\frac{4}{3}2^{2n+4}n-\frac{1}{3}2^{2n+5}n)$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

Corollary 4.9. *For $n\geq 0$, hyperbolic Cullen numbers have the following properties.*

(a) $\sum_{k=0}^{n}\mathcal{H}C_{k}=3+n-2^{n+3}+2^{n+1}n+6\times 2^{n}+j(3+n+2^{n+2}n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}C_{2k}=\frac{17}{9}+n+\frac{1}{3}2^{2n+3}n-\frac{2}{9}2^{2n+2}+j(\frac{19}{9}+n+\frac{1}{9}2^{2n+3}+\frac{1}{3}2^{2n+4}n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}C_{2k+1}=\frac{19}{9}+n+\frac{1}{3}2^{2n+4}n+\frac{1}{9}2^{2n+3}+j(\frac{17}{9}+\frac{4}{9}2^{2n+4}+n+\frac{1}{3}2^{2n+5}n)$.

We now introduce the formulas that allow us to find the sum of hyperbolic generalized Woodall numbers with negative subscripts in the following theorem.

Theorem 4.10. *For $n\geq 0$, hyperbolic generalized Woodall numbers have the following formulas:*

(a) $\sum_{k=0}^{n}\mathcal{H}W_{-k}=(-2+\frac{2}{2^{n}}-3j+n+\frac{3}{2^{n}}j+\frac{1}{2\times 2^{n}}n+jn+\frac{1}{2^{n}}jn)W_{2}+(7-\frac{7}{2^{n}}+12j-4n-\frac{11}{2^{n}}j-\frac{3}{2\times 2^{n}}n-4jn-\frac{3}{2^{n}}jn)W_{1}+(-4+\frac{5}{2^{n}}-8j+4n+\frac{8}{2^{n}}j+\frac{1}{2^{n}}n+4jn+\frac{2}{2^{n}}jn)W_{0}$.

(b) $\sum_{k=0}^{n}\mathcal{H}W_{-2k}=(-\frac{7}{9}+\frac{7}{9\times 2^{2n}}-\frac{11}{9}j+n+\frac{11}{9\times 2^{2n}}j+\frac{1}{3\times 2^{2n}}n+jn+\frac{2}{3\times 2^{2n}}jn)W_{2}+(\frac{8}{3}-\frac{8}{3\times 2^{2n}}+\frac{16}{3}j-4n-\frac{13}{3\times 2^{2n}}j-\frac{1}{2^{2n}}n-4jn-\frac{2}{2^{2n}}jn)W_{1}+(-\frac{8}{9}+\frac{17}{9\times 2^{2n}}-\frac{28}{9}j+4n+\frac{28}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+4jn+\frac{4}{3\times 2^{2n}}jn)W_{0}$.

(c) $\sum_{k=0}^{n}\mathcal{H}W_{-2k+1}=(-\frac{11}{9}+\frac{11}{9\times 2^{2n}}-\frac{7}{9}j+n+\frac{16}{9\times 2^{2n}}j+\frac{2}{3\times 2^{2n}}n+jn+\frac{4}{3\times 2^{2n}}jn)W_{2}+(\frac{16}{3}-\frac{13}{3\times 2^{2n}}+\frac{20}{3}j-4n-\frac{20}{3\times 2^{2n}}j-\frac{2}{2^{2n}}n-4jn-\frac{4}{2^{2n}}jn)W_{1}+(-\frac{28}{9}+\frac{28}{9\times 2^{2n}}-\frac{44}{9}j+4n+\frac{44}{9\times 2^{2n}}j+\frac{4}{3\times 2^{2n}}n+4jn+\frac{8}{3\times 2^{2n}}jn)W_{0}$.

Proof. It can be obtained by using Proposition 4.3.

(a) We can derive the following using the formulas in Proposition 4.3.

$$\sum_{k=0}^{n}\mathcal{H}W_{-k}=\sum_{k=0}^{n}W_{-k}+j\sum_{k=0}^{n}W_{-k+1}.$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{H}W_{-k} &= 4W_{0}(n+\frac{1}{2^{n+1}}(n+4)-\frac{1}{2^{n+2}}(n+3)-1)+2W_{1}(\frac{1}{2^{n+2}}(3n+8)-2n-\frac{1}{2^{n+1}}(3n+11)+\frac{7}{2})\\ &\quad +2W_{2}(\frac{1}{2}n+\frac{1}{2^{n+1}}(n+3)-\frac{1}{2^{n+2}}(n+2)-1)\\ &\quad +j(2W_{2}(\frac{1}{2}n+\frac{1}{2^{n}}(n+2)-\frac{1}{2^{n+1}}(n+1)-\frac{3}{2})+4W_{0}(n+\frac{1}{2^{n}}(n+3)-\frac{1}{2^{n+1}}(n+2)-2)\\ &\quad +2W_{1}(\frac{1}{2^{n+1}}(3n+5)-2n-\frac{1}{2^{n}}(3n+8)+6)).\end{aligned}$$

$$\begin{aligned}\sum_{k=0}^{n}\mathcal{H}W_{-k} &= (-2+\frac{2}{2^n}-3j+n+\frac{3}{2^n}j+\frac{1}{2\times 2^n}n+jn+\frac{1}{2^n}jn)W_2\\ &+(7-\frac{7}{2^n}+12j-4n-\frac{11}{2^n}j-\frac{3}{2\times 2^n}n-4jn-\frac{3}{2^n}jn)W_1\\ &+(-4+\frac{5}{2^n}-8j+4n+\frac{8}{2^n}j+\frac{1}{2^n}n+4jn+\frac{2}{2^n}jn)W_0.\end{aligned}$$

This proves (a). We can be prove (b) and (c) similarly way using Proposition 4.4. □

As a first special case of the above theorem, we have the following summation formulas for hyperbolic modified Woodall numbers:

Corollary 4.11. *For $n \geq 0$, hyperbolic modified Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}G_{-k} = -3+n+\frac{n+3}{2^n}+j(-3+n+\frac{2n+4}{2^n})$.

(b) $\sum_{k=0}^{n}\mathcal{H}G_{-2k} = -\frac{11}{9}+n+\frac{11+6n}{9\times 2^{2n}}+j(-\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathcal{H}G_{-2k+1} = -\frac{7}{9}+n+\frac{16+12n}{9\times 2^{2n}}+j(\frac{25}{9}+n+\frac{20+24n}{9\times 2^{2n}})$.

As a second special case of the above theorem, we have the following summation formulas for hyperbolic modified Cullen numbers:

Corollary 4.12. *For $n \geq 0$, hyperbolic modified Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}H_{-k} = 5+n-\frac{2}{2^n}+j(9-\frac{4}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}H_{-2k} = \frac{11}{3}+n-\frac{2}{3\times 2^{2n}}+j(\frac{19}{3}-\frac{4}{3\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}H_{-2k+1} = \frac{19}{3}+n-\frac{4}{3\times 2^{2n}}+j(\frac{35}{3}-\frac{8}{3\times 2^{2n}}+n)$.

As a third special case of the above theorem, we have the following summation formulas for hyperbolic Woodall numbers:

Corollary 4.13. *For $n \geq 0$, hyperbolic Woodall numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}R_{-k} = -3-n+\frac{2+n}{2^n}+j(-1-n+\frac{2+2n}{2^n})$.

(b) $\sum_{k=0}^{n}\mathcal{H}R_{-2k} = -\frac{17}{9}-n+\frac{8+6n}{9\times 2^{2n}}+j(-\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}})$.

(c) $\sum_{k=0}^{n}\mathcal{H}R_{-2k+1} = -\frac{1}{9}-n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{55}{9}-n+\frac{8+24n}{9\times 2^{2n}})$.

As a fourth special case of the above theorem, we have the following summation formulas for hyperbolic Cullen numbers:

Corollary 4.14. *For $n \geq 0$, hyperbolic Cullen numbers have the following properties:*

(a) $\sum_{k=0}^{n}\mathcal{H}C_{-k} = -1+n+\frac{2+n}{2^n}+j(1+\frac{2+2n}{2^n}+n)$.

(b) $\sum_{k=0}^{n}\mathcal{H}C_{-2k} = \frac{1}{9}+n+\frac{8+6n}{9\times 2^{2n}}+j(\frac{17}{9}+\frac{10+12n}{9\times 2^{2n}}+n)$.

(c) $\sum_{k=0}^{n}\mathcal{H}C_{-2k+1} = \frac{17}{9}+n+\frac{10+12n}{9\times 2^{2n}}+j(\frac{73}{9}+\frac{8+24n}{9\times 2^{2n}}+n)$.

5 MATRICES RELATED WITH HYPERBOLIC GENERALIZED WOODALL NUMBERS

In this section, we present matrices related with hyperbolic generalized Woodall numbers.

Now, $\{G_n\}$ defined by the third-order recurrence relation as follows

$$G_n = 5G_{n-1} - 8G_{n-2} + 4G_{n-3} \text{ with the initial conditions } G_0 = 0,\ G_1 = 1,\ G_2 = 5.$$

We present the square matrix A of order 3 as

$$A = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

such that $\det A = 1$. Then, we give the following Lemma.

Lemma 5.1. *For all integers n the following identity is true.*

$$\begin{pmatrix} \mathcal{H}W_{n+2} \\ \mathcal{H}W_{n+1} \\ \mathcal{H}W_n \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}.$$

Proof. First, we suppose that $n \geq 0$. Lemma (5.1) can be given by mathematical induction on n. If $n = 0$ we get

$$\begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^0 \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

which is true. We assume that the identity given holds for $n = k$. Thus the following identity is true.

$$\begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} = \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix}$$

For $n = k + 1$, we get

$$\begin{aligned} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^{k+1} \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}^k \begin{pmatrix} \mathcal{H}W_2 \\ \mathcal{H}W_1 \\ \mathcal{H}W_0 \end{pmatrix} \\ &= \begin{pmatrix} 5 & -8 & 4 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \begin{pmatrix} \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \\ \mathcal{H}W_k \end{pmatrix} \\ &= \begin{pmatrix} 5\mathcal{H}W_{k+2} - 8\mathcal{H}W_{k+1} + 4\mathcal{H}W_k \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix} \\ &= \begin{pmatrix} \mathcal{H}W_{k+3} \\ \mathcal{H}W_{k+2} \\ \mathcal{H}W_{k+1} \end{pmatrix}. \end{aligned}$$

If we suppose that $n < 0$ the proof can be done similarly. Consequently, by mathematical induction on n, the proof is completed. □

Note that

$$A^n = \begin{pmatrix} G_{n+1} & -8G_n + 4G_{n-1} & 4G_n \\ G_n & -8G_{n-1} + 4G_{n-2} & 4G_{n-1} \\ G_{n-1} & -8G_{n-2} + 4G_{n-3} & 4G_{n-2} \end{pmatrix}.$$

For the proof see **[**56**]**.

Theorem 5.2. *If we define the matrices $N_{\mathcal{H}W}$ and $E_{\mathcal{H}W}$ as follow,*

$$N_{\mathcal{H}W}=\begin{pmatrix}\mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0\\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1}\\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2}\end{pmatrix},\ E_{\mathcal{H}W}=\begin{pmatrix}\mathcal{H}W_{n+2} & \mathcal{H}W_{n+1} & \mathcal{H}W_n\\ \mathcal{H}W_{n+1} & \mathcal{H}W_n & \mathcal{H}W_{n-1}\\ \mathcal{H}W_n & \mathcal{H}W_{n-1} & \mathcal{H}W_{n-2}\end{pmatrix}.$$

then the following identity is true:

$$A^nN_{\mathcal{H}W}=E_{\mathcal{H}W}.$$

Proof. We can use the following identities for the proof.

$$\begin{aligned}A^nN_{\mathcal{H}W} &= \begin{pmatrix}G_{n+1} & -8G_n+4G_{n-1} & 4G_n\\ G_n & -8G_{n-1}+4G_{n-2} & 4G_{n-1}\\ G_{n-1} & -8G_{n-2}+4G_{n-3} & 4G_{n-2}\end{pmatrix}\begin{pmatrix}\mathcal{H}W_2 & \mathcal{H}W_1 & \mathcal{H}W_0\\ \mathcal{H}W_1 & \mathcal{H}W_0 & \mathcal{H}W_{-1}\\ \mathcal{H}W_0 & \mathcal{H}W_{-1} & \mathcal{H}W_{-2}\end{pmatrix},\\ &= \begin{pmatrix}b_{11} & b_{12} & b_{13}\\ b_{21} & b_{22} & b_{23}\\ b_{31} & b_{32} & b_{33}\end{pmatrix}\end{aligned}$$

where

$$\begin{aligned}
b_{11} &= \mathcal{H}W_2G_{n+1}+\mathcal{H}W_1(-8G_n+4G_{n-1})+\mathcal{H}W_0 4G_n,\\
b_{12} &= \mathcal{H}W_1G_{n+1}+\mathcal{H}W_0(-8G_n+4G_{n-1})+\mathcal{H}W_{-1}4G_n,\\
b_{13} &= \mathcal{H}W_0G_{n+1}+\mathcal{H}W_{-1}(-8G_n+4G_{n-1})+\mathcal{H}W_{-2}4G_n,\\
b_{21} &= \mathcal{H}W_2G_n+\mathcal{H}W_1(-8G_n+4G_{n-1})+\mathcal{H}W_0 4G_{n-1},\\
b_{22} &= \mathcal{H}W_1G_n+\mathcal{H}W_0(-8G_n+4G_{n-1})+\mathcal{H}W_{-1}4G_{n-1},\\
b_{23} &= \mathcal{H}W_0G_n+\mathcal{H}W_{-1}(-8G_n+4G_{n-1})+\mathcal{H}W_{-2}4G_{n-1},\\
b_{31} &= \mathcal{H}W_2G_{n-1}+\mathcal{H}W_1(-8G_n+4G_{n-1})+\mathcal{H}W_0 4G_{n-2},\\
b_{32} &= \mathcal{H}W_1G_{n-1}+\mathcal{H}W_0(-8G_n+4G_{n-1})+\mathcal{H}W_{-1}4G_{n-2},\\
b_{33} &= \mathcal{H}W_0G_{n-1}+\mathcal{H}W_{-1}(-8G_n+4G_{n-1})+\mathcal{H}W_{-2}4G_{n-2},
\end{aligned}$$

Using the Theorem (3.13) the proof is done. □

From Theorem (5.2), we can write the following corollary.

Corollary 5.3. *We have the following identity.*

(a) *If we define $N_{\mathcal{H}G}$ and $E_{\mathcal{H}G}$ as follows,*

$$N_{\mathcal{H}G}=\begin{pmatrix}\mathcal{H}G_2 & \mathcal{H}G_1 & \mathcal{H}G_0\\ \mathcal{H}G_1 & \mathcal{H}G_0 & \mathcal{H}G_{-1}\\ \mathcal{H}G_0 & \mathcal{H}G_{-1} & \mathcal{H}G_{-2}\end{pmatrix},\ E_{\mathcal{H}G}=\begin{pmatrix}\mathcal{H}G_{n+2} & \mathcal{H}G_{n+1} & \mathcal{H}G_n\\ \mathcal{H}G_{n+1} & \mathcal{H}G_n & \mathcal{H}G_{n-1}\\ \mathcal{H}G_n & \mathcal{H}G_{n-1} & \mathcal{H}G_{n-2}\end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}G}=E_{\mathcal{H}G}.$$

(b) *If we define $N_{\mathcal{H}H}$ and $E_{\mathcal{H}H}$ as follows,*

$$N_{\mathcal{H}H}=\begin{pmatrix}\mathcal{H}H_2 & \mathcal{H}H_1 & \mathcal{H}H_0\\ \mathcal{H}H_1 & \mathcal{H}H_0 & \mathcal{H}H_{-1}\\ \mathcal{H}H_0 & \mathcal{H}H_{-1} & \mathcal{H}H_{-2}\end{pmatrix},\ E_{\mathcal{H}H}=\begin{pmatrix}\mathcal{H}H_{n+2} & \mathcal{H}H_{n+1} & \mathcal{H}H_n\\ \mathcal{H}H_{n+1} & \mathcal{H}H_n & \mathcal{H}H_{n-1}\\ \mathcal{H}H_n & \mathcal{H}H_{n-1} & \mathcal{H}H_{n-2}\end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}H}=E_{\mathcal{H}H}.$$

(c) *If we define $N_{\mathcal{H}R}$ and $E_{\mathcal{H}R}$ as follows,*

$$N_{\mathcal{H}R}=\begin{pmatrix} \mathcal{H}R_2 & \mathcal{H}R_1 & \mathcal{H}R_0 \\ \mathcal{H}R_1 & \mathcal{H}R_0 & \mathcal{H}R_{-1} \\ \mathcal{H}R_0 & \mathcal{H}R_{-1} & \mathcal{H}R_{-2} \end{pmatrix},\ E_{\mathcal{H}R}=\begin{pmatrix} \mathcal{H}R_{n+2} & \mathcal{H}R_{n+1} & \mathcal{H}R_n \\ \mathcal{H}R_{n+1} & \mathcal{H}R_n & \mathcal{H}R_{n-1} \\ \mathcal{H}R_n & \mathcal{H}R_{n-1} & \mathcal{H}R_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}R}=E_{\mathcal{H}R}.$$

(d) *If we define $N_{\mathcal{H}C}$ and $E_{\mathcal{H}C}$ as follows,*

$$N_{\mathcal{H}C}=\begin{pmatrix} \mathcal{H}C_2 & \mathcal{H}C_1 & \mathcal{H}C_0 \\ \mathcal{H}C_1 & \mathcal{H}C_0 & \mathcal{H}C_{-1} \\ \mathcal{H}C_0 & \mathcal{H}C_{-1} & \mathcal{H}C_{-2} \end{pmatrix},\ E_{\mathcal{H}C}=\begin{pmatrix} \mathcal{H}C_{n+2} & \mathcal{H}C_{n+1} & \mathcal{H}C_n \\ \mathcal{H}C_{n+1} & \mathcal{H}C_n & \mathcal{H}C_{n-1} \\ \mathcal{H}C_n & \mathcal{H}C_{n-1} & \mathcal{H}C_{n-2} \end{pmatrix},$$

then we get

$$A^nN_{\mathcal{H}C}=E_{\mathcal{H}C}.$$

6 CONCLUSION

In the literature, there have been so many studies of the sequences of numbers and the sequences of numbers were widely used in many research areas, such as physics, engineering, architecture, nature and art. In this study we introduce hyperbolic generalized Woodall sequence and focused on four special cases such as hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers.

2219 In section 1, we present some important information related to generalized Woodall numbers such as reccurance relation, Binet's formula and generating function. Moreover we give some information about hyperbolic numbers and some examples studied in the literature.

2219 In section 2, we define hyperbolic generalized Woodall numbers and four special cases such as hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers. In addition, we introduce Binet's formula and generating function of hyperbolic generalized Woodall numbers and four special cases.

2219 In section 3, we define some identeties raleted to hyperbolic generalized Woodall sequence such as hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers. e.g Simpson's formula, Catalan's identity and Cassani's identity.

2219 In section 4, we define linear sum formulas related to hyperbolic generalized Woodall sequence and four special cases hyperbolic modified Woodall numbers, hyperbolic modified Cullen numbers, hyperbolic Woodall numbers and hyperbolic Cullen numbers.

2219 In section 5, we define matrix formulation to hyperbolic generalized Woodall sequence.

Linear recurrence relations (sequences) have many applications. Next, we list applications of sequences which are linear recurrence relations.

First, we present some applications of second order sequences.

- For the applications of Gaussian Fibonacci and Gaussian Lucas numbers to Pauli Fibonacci and Pauli Lucas quaternions, see [2].
- For the application of Pell Numbers to the solutions of three-dimensional difference equation systems, see [9].
- For the application of Jacobsthal numbers to special matrices, see [77].
- For the application of generalized k-order Fibonacci numbers to hybrid quaternions, see [27].
- For the applications of Fibonacci and Lucas numbers to Split Complex Bi-Periodic numbers, see [78].
- For the applications of generalized bivariate Fibonacci and Lucas polynomials to matrix polynomials, see [79].
- For the applications of generalized Fibonacci numbers to binomial sums, see [75].

- For the application of generalized Jacobsthal numbers to hyperbolic numbers, see [57].
- For the application of generalized Fibonacci numbers to dual hyperbolic numbers, see [58].
- For the application of Laplace transform and various matrix operations to the characteristic polynomial of the Fibonacci numbers, see [16].
- For the application of Generalized Fibonacci Matrices to Cryptography, see [43].
- For the application of higher order Jacobsthal numbers to quaternions, see [42].
- For the application of Fibonacci and Lucas Identities to Toeplitz-Hessenberg matrices, see [21].
- For the applications of Fibonacci numbers to lacunary statistical convergence, see [8].
- For the applications of Fibonacci numbers to lacunary statistical convergence in intuitionistic fuzzy normed linear spaces, see [33].
- For the applications of Fibonacci numbers to ideal convergence on intuitionistic fuzzy normed linear spaces, see [34].
- For the applications of k-Fibonacci and $k-$Lucas numbers to spinors, see [35].
- For the application of dual-generalized complex Fibonacci and Lucas numbers to Quaternions, see [73].
- For the application of special cases of Horadam numbers to Neutrosophic analysis see [23].
- For the application of Hyperbolic Fibonacci numbers to Quaternions, see [14].

We now present some applications of third order sequences.

- For the applications of third order Jacobsthal numbers and Tribonacci numbers to quaternions, see [11] and [10], respectively.
- For the application of Tribonacci numbers to special matrices, see [80].
- For the applications of Padovan numbers and Tribonacci numbers to coding theory, see [45] and [4], respectively.
- For the application of Pell-Padovan numbers to groups, see [15].
- For the application of adjusted Jacobsthal-Padovan numbers to the exact solutions of some difference equations, see [22].
- For the application of Gaussian Tribonacci numbers to various graphs, see [72].
- For the application of third-order Jacobsthal numbers to hyperbolic numbers, see [18].
- For the application of Narayan numbers to finite groups see [36].
- For the application of generalized Guglielmo numbers to Gaussian numbers, see [54].
- For the application of generalized Woodall numbers to Gaussian numbers, see [55].
- For the application of generalized third-order Jacobsthal sequence to binomial transform, see [59].
- For the application of generalized Generalized Padovan numbers to Binomial Transform, see [60].
- For the application of generalized Tribonacci numbers to Gaussian numbers, see [61].
- For the application of generalized Tribonacci numbers to Sedenions, see [62].
- For the application of Tribonacci and Tribonacci-Lucas numbers to matrices, see [63].
- For the application of generalized Tribonacci numbers to circulant matrix, see [64].
- For the application of Tribonacci and Tribonacci-Lucas numbers to hybrinomials, see [76].
- For the application of hyperbolic Leonardo and hyperbolic Francois numbers to quaternions, see [19].

Next, we now list some applications of fourth order sequences.

- For the application of Tetranacci and Tetranacci-Lucas numbers to quaternions, see [65].
- For the application of generalized Tetranacci numbers to Gaussian numbers, see [66].
- For the application of Tetranacci and Tetranacci-Lucas numbers to matrices, see [67].
- For the application of generalized Tetranacci numbers to binomial transform, see [68].

We now present some applications of fifth order sequences.

- For the application of Pentanacci numbers to matrices, see [46].
- For the application of generalized Pentanacci numbers to quaternions, see [49].

- For the application of generalized Pentanacci numbers to binomial transform, see [50].

  We now present some applications of second order sequences of polynomials.

- For the application of generalized Fibonacci Polynomials to the summation formulas, see [70].

- For some applications of generalized Fibonacci Polynomials, see [71].

  We now present some applications of third order sequences of polynomials.

- For some applications of generalized Tribonacci Polynomials, see [69].

COMPETING INTERESTS

Authors have declared that no competing interests exist.

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