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Existence of Entropy Solution for Degenerate Parabolic-Hyperbolic Problem Involving *p*(*x*)-Laplacian with Neumann Boundary Condition

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Abstract

We consider a strongly non-linear degenerate parabolic-hyperbolic problem with *p*(*x*)-Laplacian diffusion flux function. We propose an entropy formulation and prove the existence of an entropy solution.

Keywords

Lebesgue and Sobolev Spaces with Variable Exponent, Weak Solution, Entropy Solution, Degenerate Parabolic-Hyperbolic Equation, Conservation Law, Leray Lions Type Operator, Neumann Boundary Condition, Existence Result

1. Introduction

In this paper, we consider the following non-linear degenerate parabolic-hyperbolic problem:

$$(\mathrm{P})\quad \begin{cases} u_t - \operatorname{div}\left(\left|\nabla\phi(u)\right|^{p(x)-2}\nabla\phi(u) - f(u)\right) = 0 & \text{in } Q, \\ u(0,x) = u_0(x) & \text{in } \Omega, \\ \left(f(u) - \left|\nabla\phi(u)\right|^{p(x)-2}\nabla\phi(u)\right)\cdot\eta = 0 & \text{on } \Sigma \end{cases}$$

where $Q = (0,T)\times\Omega$ and $\Sigma = (0,T)\times\partial\Omega$. Here Ω is a smooth bounded open domain in $\mathbb{R}^N$ with smooth boundary $\partial\Omega$, $\eta(x)$ the unit normal to $\partial\Omega$ outward to Ω and $u:(t,x)\in Q\to\mathbb{R}$ is the unknown function, $T>0$ is a fixed time. The initial data u_0 is assumed to be bounded measurable. This mean

$$u_0 \in L^\infty(\Omega). \tag{1.1}$$

We assume that the convection flux:

$$f : \mathbb{R} \to \mathbb{R}^N \text{ is continuous.} \tag{1.2}$$

Moreover, we suppose that $[0, u_{\max}]$ where $u_{\max} > 0$ will be an invariant domain of the solution of **(P)** and then

$$f(0) = f(u_{\max}) = 0. \tag{1.3}$$

With hypothesis (1.3), according to (1.1), one can take $u_0 \in [0, u_{\max}]$ (see [1]-[3]). As in [2], the diffusion flux function φ is continuous and nondecreasing assumed to be constant on certain interval of values of *u*. There exists a closed set $E \subset [0, u_{\max}]$ such that ϕ is strictly increasing on $[0, u_{\max}] \setminus E$, and the Lebesgue measure of $\phi(E)$ is zero.

Our non-linear partial differential equations includes the particular hyperbolic conservation law. The only notion of weak solution do not leads to well-possessedness and we need an entropy formulation (see [4] [5]).

The function $p : \bar{\Omega} \to \mathbb{R}$ is a continuous function. The associated operator $w \to -\text{div}\left(|\nabla w|^{p(x)-2} \nabla w\right)$ is a prototype of Leray-Lions operator acting from $W_0^{1,p(x)} \to W^{1,p'(x)}$ with $p'(x) = \dfrac{p(x)}{p(x)-1}$ and $p(x) > 1$. The variable exponent *p* depend on the space variable *x*. The particular case where $p(x) \equiv 2$ was treated in [2]. The interest motivation of the study of this kind of problem is due to the fact that they can model various phenomena which arise in the study of elastic mechanic (see [6]), electro-rheological fluids (see [7]) or image restoration (see [8]).

We propose an entropy formulation for **(P)**. This entropy formulation generalizes the notion of entropy solution of [2]. In this entropy formulation, the boundary condition is taken in a weak sense, which makes it easy to overcome difficulties in the treatment of the boundary condition. For the proof of the existence of an entropy solution, we approach the problem **(P)** by regularizing the data so that the approximate problem is non-degenerate. Thanks to a priori estimations, we show that the sequence of solutions converges towards an entropy process solution which coincides with the entropy solution.

This article consists of four additional sections. In the second section, we introduce some basic properties of the generalized Lebesgue-Sobolev spaces with variable exponent. In section 3, we propose an entropy formulation for problem **(P)** and prove existence in section 4. We end with a conclusion and perspectives.

2. Lebesgue and Sobolev Space with Variable Exponent

This section is devoted to basic property of Lebesgue and Sobolev spaces with variable exponent, that depend on *x*. Let us recall some elementary properties:

The measurable function

$$p(.) : \bar{\Omega} \to \mathbb{R} \text{ such that } 1 < p_- \le p \le p_+ < +\infty \tag{1.4}$$

where

$$p_- = \operatorname{essinf}_{x \in \bar{\Omega}} p(x);\ p_+ = \operatorname{esssup}_{x \in \bar{\Omega}} p(x). \tag{1.5}$$

We define the Lebesgue space with variable exponent $L^{p(.)}(\Omega)$ as the set of all measurable functions $u : \Omega \to \mathbb{R}$ for which the convex modular

$$\rho_{p(.)}(u) := \int_\Omega |u(x)|^{p(x)} \mathrm{d}x \le \infty.$$

If the exponent is bounded, *i.e.*, if $p_+ < +\infty$, then the expression

$$|u|_{p(.)} = \inf \left\{ \lambda > 0 : \rho_{p(.)}\left(\frac{u}{\lambda}\right) \le 1 \right\}$$

defines a norm in $L^{p(.)}(\Omega)$, called the Luxembourg norm.

The space $\left(L^{p(.)}(\Omega), |.|_{p(.)}\right)$ is a separable Banach space. Moreover, if $1 < p_- \le p_+ < +\infty$, then $L^{p(.)}(\Omega)$ is uniformly convex, hence reflexive, and its dual space is isomorphic to $L^{p'(.)}(\Omega)$, where $p'(x) = \frac{p(x)}{p(x)-1}$ is a conjugate exponent of $p(x)$.

With exponent variable, we have a kind of Hölder type inequality:

$$\left|\int_\Omega uv\mathrm{d}x\right| \le \left(\frac{1}{p_-} + \frac{1}{p'_-}\right)|u|_{p(.)}|v|_{p'(.)} \quad \forall (u,v) \in L^{p(.)}(\Omega) \times L^{p'(.)}(\Omega).$$

Let

$$W^{1,p(x)}(\Omega) = \left\{u \in L^{p(.)}(\Omega); \nabla u \in L^{p(.)}(\Omega)\right\} \tag{1.6}$$

which is a Banach space equipped with the following norm

$$\|u\|_{1,p(.)} = |u|_{p(.)} + \|\nabla u\|_{p(.)}. \tag{1.7}$$

The space $\left(W^{1,p(.)}(\Omega), \|.\|_{1,p(.)}\right)$ is a separable and reflexive Banach space.

3. Entropy Formulation

3.1. Definition of Entropy Solution

Definition 3.1

A measurable function $u \in L^\infty(Q)$ is weak solution of **(P)** if for $\phi(u) \in L^\infty\left(0,T; W^{1,p(x)}(\Omega)\right)$, $\xi(t,x) \in L^1\left((0,T), W^{1,p(x)}(\Omega)\right) \cap L^\infty(Q)$ such that $\xi_t \in L^1(Q)$

$$\begin{aligned}&\int_0^T \int_\Omega u\xi_t \mathrm{d}x\mathrm{d}t + \int_0^T \int_\Omega \left(f(u) - |\nabla\phi(u)|^{p(x)-2} \nabla\phi(u)\right) \cdot \nabla\xi \mathrm{d}x\mathrm{d}t \\ &+ \int_\Omega u_0 \xi(0,x)\mathrm{d}x = 0.\end{aligned} \tag{1.8}$$

Definition 3.2

A weak solution is called entropy solution of **(P)** if: $u \in [0, u_{\max}]$, for $\xi(t,x) \in L^1\left(0,T; W^{1,p(x)}(\Omega)\right) \cap L^\infty(Q)$; $\phi(u) \in L^\infty\left(0,T; W^{1,p(x)}(\Omega)\right)$ $k \in [0, u_{\max}]$, the following inequality holds

$$\begin{aligned}&\int_0^T\int_\Omega \operatorname{sign}(u-k)\Big(f(u)-f(k)-\left|\nabla\phi(u)\right|^{p(x)-2}\nabla\phi(u)\Big)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t\\&+\int_0^T\int_\Omega|u-k|\xi_t\,\mathrm{d}x\mathrm{d}t+\int_\Omega|u_0-k|\xi(0,x)\,\mathrm{d}x\\&+\int_0^T\int_{\partial\Omega}\left|f(k)\cdot\eta(x)\right|\xi\,\mathrm{d}\mathcal{H}^{\ell-1}(x)\,\mathrm{d}t\ge 0.\end{aligned}\tag{1.9}$$

Remark 3.3

Notice that if $\phi(u)\in L^\infty\left(0,T;W^{1,p(x)}(\Omega)\right)$ then

$$\left|\nabla\phi(u)\right|^{p(x)-2}\nabla\phi(u)\in L^1\left(0,T,L^{p'(x)}(\Omega)\right).\tag{1.10}$$

3.2. Entropy Process Solution

In this subsection, let us introduce a notion of entropy process solution based upon the so-called "nonlinear L^∞ weak $\star$ convergence" property, which is well-known in the equivalent framework of the notion of measure-valued solution developed earlier by Tartar and Diperna (see [9]).

Definition 3.4

A measurable bounded function $\mu:(0,1)\times Q\to[0,u_{\max}]$ is called entropy process solution of evolution problem **(P)** if for $\phi(u)\in L^\infty\left(0,T;W^{1,p(x)}(\Omega)\right)$, $\xi(t,x)\in L^1\left(0,T;W^{1,p(x)}(\Omega)\right)\cap L^\infty(\Omega)$, $k\in[0,u_{\max}]$

$$\begin{aligned}&\int_0^1\int_0^T\int_\Omega \operatorname{sign}(\mu-k)\Big(f(\mu)-f(k)-\left|\nabla\phi(\mu)\right|^{p(x)-2}\nabla\phi(\mu)\Big)\cdot\nabla\xi\,\mathrm{d}x\mathrm{d}t\mathrm{d}\alpha\\&+\int_0^1\int_0^T\int_\Omega|\mu-k|\xi_t\,\mathrm{d}x\mathrm{d}t\mathrm{d}\alpha+\int_0^1\int_\Omega|u_0-k|\xi(0,x)\,\mathrm{d}x\mathrm{d}\alpha\\&+\int_0^T\int_{\partial\Omega}\left|f(k)\cdot\eta(x)\right|\xi\,\mathrm{d}\mathcal{H}^{\ell-1}(x)\,\mathrm{d}t\ge 0.\end{aligned}\tag{1.11}$$

Remark 3.5

$\mu\in L^\infty\left((0,1)\times Q\right)$ is referred to as the "process function"; it is related to the distribution function of the Young measure.

We have only considered α-independent data u_0. In this case, the notion of entropy process solution is just a technical tool that permits to bypass the lack of strong compactness of sequences of approximate solutions.

4. Existence of Entropy Solution

This main result is the following problem:

Theorem 4.1

Assume that (1.1), (1.2), (1.3) and (1.4) holds. There exists an entropy process solution to **(P)**.

Proof

Contrarily to [2] due to the strong non-linearity and the presence of $p(x)$-Laplacian operator, it seem difficult to apply the viscosity approximation but we can approximate problem **(P)** by regularized f and ϕ by a family of sequence f_ϵ and ϕ_ϵ such that f_ϵ converge to f uniformly on compact set as $\epsilon\to 0$ and ϕ_ϵ converges to ϕ in $H^1(\Omega)$. Let $u_0^\epsilon\to u_0$ almost everywhere. Then, refer to [10] there exists a weak solution $u_\epsilon\in L^{p(x)}\left(0,T;W^{1,p(x)}(\Omega)\right)$ in the following sense

$$\partial_t u_\epsilon + \operatorname{div} f_\epsilon(u_\epsilon) = \operatorname{div}\left(\left|\nabla\phi_\epsilon(u_\epsilon)\right|^{p(x)-2}\nabla\phi_\epsilon(u_\epsilon)\right) + \epsilon\nabla u_\epsilon + s \quad (1.12)$$

where $s \in L^\infty(Q)$ a source term. By technique of doubling the time variable, we obtained a L^1 contraction property and comparison principle for weak solution of regularized problem. Moreover u_ϵ verifies the entropy inequality with f_ϵ and ϕ_ϵ.

From now, we have that the following quantities are uniformly bounded in ϵ: $\|u_\epsilon\|_{L^\infty}$; $\|\phi_\epsilon\|_{L^1\left(0,T;W^{1,p(x)}(\Omega)\right)}$, the time and space translate of ϕ_ϵ in L^1. Indeed, let:

$$L(t) = \|u_0\|_{L^\infty} + \int_0^t \|s(\tau,.)\|_{L^\infty}\,\mathrm{d}\tau .$$

It is easy to see that the function L is a solution of regularized problem with x-constant data $\|u_0\|_{L^\infty}$, $\|s(t,.)\|_{L^\infty}$. The comparison principle mentioned ensures that a.e. on Q

$$-L(T) \le -L(t) \le u_\epsilon(t,x) \le L(t) \le L(T).$$

Next, we use $\phi_\epsilon(u_\epsilon)$ as a test function in (1.12). The product between $\frac{\partial}{\partial t}u_\epsilon$ and $\phi_\epsilon(u_\epsilon)$ is handled using the usual chain rule argument (see, e.g. [11]) can be adapted to space $L^{p(x)}$, where the relevant duality is between the space $X := L^{p(x)}\left(0,T;W_0^{1,p(x)}(\Omega)\right) \cap L^\infty(Q)$ and the space $L^{p'(x)}\left(0,T;W^{-1,p'(x)}(\Omega)\right) + L^1(Q) \subset X^\star$. Here we are also exploiting the L^∞ bound on $f_\epsilon(u_\epsilon)$ in a straightforward fashion to treat the term $f_\epsilon(u_\epsilon)\cdot\nabla\phi_\epsilon(u_\epsilon)$; but notice that using the Green Gauss trick (1.21) below, we can supply a finer analysis of this term.

For the space translate estimate, we first use (1.12) to get, for a.e. $t, t+\delta t \in (0,T)$

$$\begin{aligned} &\int_\Omega \left(u_\epsilon(t+\delta t) - u_\epsilon(t)\right)\xi \\ &= \int_t^{t+\delta t}\int_\Omega \left(-f_\epsilon(u_\epsilon) + \nabla\left|\phi_\epsilon(u_\epsilon)\right|^{p(x)-2}\nabla\phi_\epsilon(u_\epsilon)\right)\cdot\nabla\xi + \int_t^{t+\delta t}\int_\Omega \epsilon\nabla u_\epsilon \xi. \end{aligned} \quad (1.13)$$

Taking $\xi = \phi_\epsilon(u_\epsilon)(t+\delta t) - \phi_\epsilon(u_\epsilon)(t)$ and integrating in t, using the two previously obtained estimates, we deduce that

$$\iint_Q \left|u_\epsilon(t+\delta t) - u_\epsilon(t)\right|\left|\phi_\epsilon(u_\epsilon)(t+\delta t) - \phi_\epsilon(u_\epsilon)(t)\right| \le C|\delta t|. \quad (1.14)$$

Now, let W be a common for all ϵ concave modulus of continuity for ϕ_ϵ on $[-L(T), L(T)]$ and Π be its inverse. Set $\bar{\Pi}(a) = a\Pi(a)$. Let $\bar{W}$ be a inverse of $\bar{\Pi}$. One can see that $\bar{W}$ is concave, continuous and $\bar{W}(0) = 0$. Set $y(t,x) = u_\epsilon(t+\delta t, x)$ and $z(t,x) = u_\epsilon(t,x)$, such that $\phi_\epsilon(u_\epsilon)(.,.) \in C^\infty(Q)$

$$\begin{aligned} J &= \int_Q \left|\phi_\epsilon(y) - \phi_\epsilon(z)\right| \\ &= \int_Q \left|\bar{W}\left(\bar{\Pi}\left(\left|\phi_\epsilon(y) - \phi_\epsilon(z)\right|\right)\right)\right| |\Omega| T \bar{W}\left(\frac{1}{|\Omega| T}\int_K \bar{\Pi}\left(\left|\phi_\epsilon(y) - \phi_\epsilon(z)\right|\right)\right). \end{aligned}$$

Since $\left|\phi_\epsilon(y) - \phi_\epsilon(z)\right| \le W(|y-z|)$ we have

$$\Pi\left(\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|\right)\le|y-z| \quad \text{and}$$
$$\bar{\Pi}\left(\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|\right)=\Pi\left(\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|\right)\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|\le|y-z|\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|.$$

Therefore (1.14) implies

$$\int\int_Q\left|\phi_\epsilon(u_\epsilon)(t+\delta t)-\phi_\epsilon(u_\epsilon)(t)\right|$$
$$\le|K|\bar{W}\left(\frac{1}{|K|}\int_Q|y-z|\left|\phi_\epsilon(y)-\phi_\epsilon(z)\right|\right)\le Cw_{\phi_\epsilon}(\delta)$$

where $w_\phi\in C\left(\mathbb{R}^+,\mathbb{R}^+\right)$, $w_\phi(0)=0$.

Thanks to these all estimates and standard compactness results, there exists a (not labelled) sequence $\epsilon\to 0$ such that:

$w_\epsilon=\phi_\epsilon(u_\epsilon)$ converges strongly in $L^1(Q)$ and pointwise a.e. on Q;

∇w_ϵ converges weakly in $L^{p(x)}(Q)$;

$\left|\nabla w_\epsilon\right|^{p(x)-2}\nabla w_\epsilon$ converges weakly in $L^{p(x)}(Q)$ to some limit χ;

u_ϵ converges to $\mu:Q\times(0,1)\to\mathbb{R}$ in the sense of L^∞-weak star.

Let us introduce the function

$$u(t,x)=\int_0^1\mu(t,x,\alpha)\mathrm{d}\alpha,\ \text{for a.e. } (t,x)\in Q. \tag{1.15}$$

Thanks to the convergence of ϕ_ϵ to ϕ, we can identify the limit of $w_\epsilon(.,.)$ with $\int_0^1\phi\left(\mu(.,.,\alpha)\right)\mathrm{d}\alpha$. Moreover, since w_ϵ is converging strongly, $\phi\left(\mu(.,.,\alpha)\right)$ is actually independent of $\alpha\in(0,1)$ and equals $\phi\left(u(.,.)\right)$. Using distributional derivatives, we also identify the limit of ∇w_ϵ with $\nabla\phi(u)$.

We have now come to the main step of the proof of this Theorem, namely to improve the weak convergence of $\nabla\phi_\epsilon(u_\epsilon)$ to strong convergence, and to identify the weak limit of $\left|\nabla\phi_\epsilon(u_\epsilon)\right|^{p(x)-2}\nabla\phi_\epsilon(u_\epsilon)$ with $\left|\nabla\phi(u)\right|^{p(x)-2}\nabla\phi(u)$, where u is defined in (1.15); of course, the chief difficulty comes from the lack of strong convergence of u_ϵ.

We begin by specifying the test function in (1.12) as $w_\epsilon\zeta$, yielding

$$\begin{aligned}&\int_0^T\left\langle\frac{\partial}{\partial t}u_\epsilon,w_\epsilon\zeta\right\rangle-\int_Q f(u_\epsilon)\cdot\nabla(w_\epsilon\zeta)\\&+\int_Q\left|\nabla\phi_\epsilon(u_\epsilon)\right|^{p(x)-2}\nabla\phi_\epsilon(u_\epsilon)\cdot\nabla(w_\epsilon\zeta)-\int_Q sw_\epsilon\zeta=0\end{aligned} \tag{1.16}$$

where $w_\epsilon=\phi_\epsilon(u_\epsilon)$ and $\zeta\in D\left([0,T)\right)$ is nonincreasing with $\zeta(0)=1$. Denote by $I_{i,\epsilon}$, the integral in the left hand side. Next, we pass to the limit into the weak formulation (1.12), obtaining

$$\begin{aligned}&\partial_t u_\epsilon+\mathrm{div}\int_0^1 f(\mu)\mathrm{d}\alpha=\mathrm{div}\chi+s \text{ in } L^{p'(x)}\left(0,T;W^{-1,p'(x)}(\Omega)\right)+L^1(Q)\\&u_\epsilon(0,x)=u_0(x).\end{aligned} \tag{1.17}$$

In (1.17), we take $w\zeta$ as test function, where $w=\phi(u)$, u is defined in (1.15), and ζ is as specified above. The result is

$$\int_0^T\left\langle\frac{\partial}{\partial t}u,w\zeta\right\rangle-\int_Q f(\mu)\cdot\nabla(w\zeta)+\int_Q\chi\cdot\nabla(w\zeta)-\int_Q sw\zeta=0. \tag{1.18}$$

Denote by I_i $i=1,2,3,4$ the integrals in the left hand side of (1.18)

$$I_3 \geq \liminf_{\epsilon \to 0} \int_Q |\nabla \phi|^{p(x)}. \tag{1.19}$$

A crucial role is played by the following calculation, which reveals that the lack of strong convergence of $f_\epsilon(u_\epsilon)$ is not an obstacle. Indeed,

$$\begin{aligned} \int_Q \int_0^1 f(\mu)\cdot \nabla \phi(u) &= \int_Q \int_0^1 f(\mu)\cdot \nabla \phi(\mu) \\ &= \int_0^1 \int_0^T \int_\Omega \operatorname{div}\left(\int_0^\mu f(s)\mathrm{d}\phi(s)\right) \\ &= \int_0^T \int_{\partial\Omega} \int_0^\mu f(s)\mathrm{d}\phi(s)\cdot \eta = 0. \end{aligned} \tag{1.20}$$

Because for a.e $\alpha \in (0,1)$ we have $\phi(\mu) = \phi(u)$ in $L^{p(x)}\left(0,T;W^{1,p(x)}(\Omega)\right)$. By similar (simpler) arguments and $u_\epsilon \in L^{p(x)}\left(0,T;W^{1,p(x)}(\Omega)\right)$, we also have

$$\int_Q f_\epsilon(u_\epsilon)\cdot \nabla \phi_\epsilon(u_\epsilon) = \int_0^T \int_\Omega \operatorname{div}\left(\int_0^{u_\epsilon} f_\epsilon(s)\mathrm{d}\phi_\epsilon(s)\right) = 0. \tag{1.21}$$

Consequently, we can make I_2 and $I_{2,\epsilon}$ (for each $\epsilon > 0$) vanish.

$$\begin{aligned} I_1 &= \int_0^T \left\langle \frac{\partial}{\partial t} u, \phi(u)\zeta \right\rangle \\ &= -\int_Q \left(\int_0^u \phi(s)\mathrm{d}s\right)\zeta' - \int_\Omega \left(\int_0^{u_0} \phi(s)\mathrm{d}s\right) \\ &= \int_Q \left(\int_0^{\int_0^1 \mu(t,x,\alpha \mathrm{d}\alpha)} \phi(s)\mathrm{d}s\right)(-\zeta') - \int_\Omega \left(\int_0^{u_0} \phi(s)\mathrm{d}s\right) \\ &\leq \int_Q \left(\int_0^1 \int_0^\mu \phi(s)\mathrm{d}s\right)(-\zeta') - \int_\Omega \left(\int_0^{u_0} \phi(s)\mathrm{d}s\right) \\ &= \lim_{\epsilon \to 0} \left(-\int_Q \left(\int_0^{u_\epsilon} \phi_\epsilon(s)\mathrm{d}s\right)\zeta' - \int_\Omega \left(\int_0^{u_0} \phi_\epsilon(s)\mathrm{d}s\right)\right) \\ &= \lim_{\epsilon \to 0} \int_0^T \left\langle \frac{\partial}{\partial t} u_\epsilon, \phi_\epsilon(u_\epsilon)\zeta \right\rangle = \lim_{\epsilon \to 0} I_{1,\epsilon}. \end{aligned}$$

Here we have use the fact that the convex function $\int_0^z \phi(s)\mathrm{d}s$ converge uniformly on any compact set of $\mathbb{R}$ to $\int_0^z \phi_\epsilon(s)\mathrm{d}s$, due to Jensen's inequality. Then, we have $I_1 \leq I_{1,\epsilon}$.

It is clear that $I_{3,\epsilon} \to I_3$ as $\epsilon \to 0$. Letting ζ tend to $\mathbb{1}[0,T)$, the desired inequality (1.19) follows from subtracting the $\epsilon \to 0$ limit of (1.16) from (1.18) and the above calculations. From (1.19)

$$|\nabla w_\epsilon|^{p(x)-2} \nabla w_\epsilon - |\nabla \phi(u)|^{p(x)-2} \nabla \phi(u) \to 0 \tag{1.22}$$

weakly in $L^{p'(x)}$ as $\epsilon \to 0$ Hence $\chi = |\nabla \phi(u)|^{p(x)-2} \nabla \phi(u)$.

Simultaneously, from the strict monotonicity of $|r|^{p(x)-2} r$ we deduce that, firstly, the convergence in (1.22) also takes place a.e. in *Q*; secondly, that (1.19) actually holds with an equality sign. Next

$$|\nabla w_\epsilon|^{p(x)} \to |\nabla \phi(u)|^{p(x)} \text{ a.e in } Q; \int_Q |\nabla w_\epsilon|^{p(x)} \to \int_Q |\nabla \phi_\epsilon|^{p(x)} \text{ as } \epsilon \to 0$$

Hence, we deduce that a subsequence of $\left(|\nabla w_\epsilon|^{p(x)}\right)_\epsilon$ converges to $|\nabla \phi_\epsilon|^{p(x)}$ strongly in $L^1(Q)$. By Vitali theorem yields the strong $L^{p(x)}$ convergence of ∇w_ϵ, along a subsequence if necessary, to a limit already identified as ∇w, $w = \phi(u)$. Finally, uses the continuity of entropy fluxes and non nonlinear L^∞

weak-$\star$ convergence we can pass to the limit in the entropy inequalities corresponding to ϵ data and deduce that μ is an entropy process solution.

From now, it remains to prove that entropy process solution is equivalent to entropy solution.

Theorem 4.2

Suppose all assumptions (1.1), (1.2), (1.3) and (1.4) holds. Let μ be an entropy process solution of the problem **(P)** with initial data u_0. Then it is unique. Moreover, there exists a function $u \in L^{\infty}(Q)$ such that $\mu(t,x,\alpha) = u(t,x)$ for a.e. $(t,x,\alpha) \in Q \times (0,1)$.

Proof (Sketched)

The uniqueness of an entropy process solution can be established using Kruzhkov's method, along the lines of Carrillo. In fact, taking two entropy process solutions $\mu(t,x,\alpha)$ and $\mu(t,x,\beta)$ for $(t,x) \in Q$, and $(\alpha,\beta) \in (0,1)^2$ with the choice of an appropriate test function we can deduce uniqueness and that it is α–independent this mean that $\mu(t,x,\alpha) = u(t,x)$ for $(t,x,\alpha) \in Q \times (0,1)$ and u is an entropy solution of **(P)**.

5. Conclusion and Perspective of Uniqueness of Entropy Solution

In this paper, it is a question of proposing an entropy formulation of the problem **(P)** and proving the existence of a solution. The approach to achieve this is different from that used in [2] and also in [3]. We take advantage of the L^{∞} bound of the sequence of solutions and some a priori estimates to show that the sequence of approximate solutions converges towards a notion of solution called entropy process solution and this notion coincides with the notion of entropy solution.

The question of uniqueness deserves to be looked at. Two difficulties may appear: first, the doubling of variables method (see [4]) is not adapted because of the presence of $p(x)$. Then it is difficult as in the papers [2] [3] to prove that entropy solution is trace regular.

It is possible to study trace regularity of solution of the stationary problem associated with **(P)** and to use the arguments of nonlinear semigroup theory.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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