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Uniform Lipschitz Bound for a Competition Diffusion Advection System with Strong Competition

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Abstract

We prove the uniform Lipschitz bound of solutions for a nonlinear elliptic system modeling the steady state of populations that compete in a heterogeneous environment. This extends known quasi-optimal regularity results and covers the optimal case for this problem. The proof relies upon the blow-up technique and the almost monotonicity formula by Caffarelli, Jerison and Kenig.

Keywords

Diffusion-Advection System, Free Boundary Problem, Uniform Lipschitz Bound

1. Introduction

In this paper, we consider the following competition-diffusion-advection system

$$\begin{cases} u_t - \nabla\cdot\left[\mu_1\nabla u - \alpha u\nabla m\right] = (m - bu)u - kuv & \text{in } \Omega\times(0,\infty), \\ v_t - \nabla\cdot\left[\mu_2\nabla v - \beta v\nabla m\right] = (m - cv)v - kuv & \text{in } \Omega\times(0,\infty), \\ \mu_1\dfrac{\partial u}{\partial \nu} - \alpha u\dfrac{\partial m}{\partial \nu} = \mu_2\dfrac{\partial v}{\partial \nu} - \beta v\dfrac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega\times(0,\infty), \end{cases} \tag{1}$$

where $u = u(x,t)$ and $v = v(x,t)$ denote the densities of two competing species at position $x\in\Omega$ and time $t\geq 0$. $m = m(x)$ represents a local population growth rate that depends on location. In some sense, $m(x)$ can reflect the quality and quantity of resources available at the location *x*, where the favorable region $\{x\in\Omega : m(x) > 0\}$ acts as a source and the unfavorable part $\{x\in\Omega : m(x) < 0\}$ is a sink region [1]. $\mu_1\nabla u$ and $\mu_2\nabla v$ account for random diffusion, and $\alpha u\nabla m$ and $\beta v\nabla m$ represent movement upward along the environmental gradient. The two non-negative constants α and β measure the tendency of the two species to move up along the gradient of $m(x)$, and μ_1 and μ_2 represent the random diffusion rates of two species, respectively. The positive constants b and c are the intraspecific and k the interspecific competition rates. Ω is a bounded domain in $\mathbb{R}^N\,(N \ge 1)$ with smooth boundary $\partial\Omega$. The Zero-flux boundary condition in (1) means that no individuals cross the boundary of the habitat.

From the mathematical viewpoint, qualitative properties of non-negative solutions of system (1) have been extensively studied. We will briefly review some of them, for a more complete and detailed discussion, see [2]. For the case when $k=1$, $\alpha > 0, \beta \equiv 0$, Cantrell *et al.* [3] [4] showed that if Ω is convex and $\mu_1 = \mu_2$, then for positive small α the semi-trivial steady state $(\theta(x,\alpha,\mu),0)$ of (1) is globally asymptotically stable. In contrast, Cantrell *et al.* [3] and Chen *et al.* [5] proved that for large values α system (1) can have a stable positive steady state and two competing species coexist for large α. For the case when $k=1$, $\alpha > 0, \beta > 0$, Chen *et al.* [1] showed that if the ratio β/μ_2 is suitable related, then the two species coexist for sufficiently large α.

For the case when k is sufficiently large, we proved in [6] that system is expected to approach a limiting configuration where all the populations survive but have disjoint habitats. Precisely, we proved that k-dependent solutions $\{(u_k, v_k)\}$ of (1) are uniformly bounded in Hölder spaces and they converge to the positive and negative parts of a solution of a scalar limit problem. The objective of this paper is to improve the result in [6], proving the uniform bound in Lipschitz norm. Without loss of generality, we set $\mu_1 = \mu_2 = 1$ in system (1), and consider the time-independent case:

$$\begin{cases} -\nabla\cdot[\nabla u_k - \alpha u_k \nabla m] = (m - bu_k)u_k - ku_k v_k & \text{in } \Omega, \\ -\nabla\cdot[\nabla v_k - \beta v_k \nabla m] = (m - cv_k)v_k - ku_k v_k & \text{in } \Omega, \\ \dfrac{\partial u_k}{\partial \nu} - \alpha u_k \dfrac{\partial m}{\partial \nu} = \dfrac{\partial v_k}{\partial \nu} - \beta v_k \dfrac{\partial m}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{2}$$

Throughout this paper, we assume that the function $m(x) \in C^2(\bar{\Omega})$ and $m(x)$ is positive somewhere in Ω. Our main result is as follows.

Theorem 1. *Let* $\{(u_k, v_k)\}$ *be non-negative solutions of* (2). *Then for every compact set* $\Omega' \subset\subset \Omega$ *there exist* $M > 0$ *independent of k such that*

$$\|(u_k, v_k)\|_{Lip(\Omega')} := \|(u_k, v_k)\|_{L^\infty(\Omega')} + \|\nabla(u_k, v_k)\|_{L^\infty(\Omega')} \le M.$$

Note that the study of strong-competition limits in corresponding elliptic or parabolic systems is of interest not only for questions of spatial segregation and coexistence, in population dynamics, as here and in [7]-[14], but also is key to the understanding of phase separation of Gross-Pitaevskii systems of modeling Bose-Einstein condensates, see [15]-[24] and reference therein.

The uniform Hölder regularity in related problems have been studied by many authors, see [10] [11] [15] [25], for the elliptic case, [15] [18] for the parabolic case, and [26] [27] [28] for the fractional diffusion case. Concerning the uniform Lipschitz boundedness, some results have already been observed in literature. For the case of two components without advection and reaction terms, Conti, Terracini and Verzini in [10] proved that if $\{(u_k, v_k)\} \in H^1(\Omega)$ are non-negative solutions of

$$\begin{cases} \Delta u_k = k u_k v_k & \text{in } \Omega, \\ \Delta v_k = \gamma k u_k v_k & \text{in } \Omega, \\ u_k = \phi, v_k = \psi & \text{on } \partial\Omega \end{cases}$$

with $\gamma > 0$ and traces $\phi, \psi \in Lip(\partial\Omega)$, then $\{(u_k, v_k)\}$ is uniformly bounded in the Lipschitz norm. By using Kato's inequality, Wang and Zhang [14] generalized the result to arbitrary number of components (possibly with suitable reaction terms). In [29] Berestycki, Lin, Wei and Zhao deal with the Gross-Pitaevskii system in dimension $N = 1$, they proved that if $\{(u_k, v_k)\} \in H_0^1([0,1])$ are uniformly L^∞ bounded solutions of

$$\begin{cases} -u_k'' + \lambda_{1,k} u_k = \omega_1 u_k^3 - k u_k v_k^2 & \text{in } [0,1], \\ -v_k'' + \lambda_{2,k} u_k = \omega_2 v_k^3 - k v_k u_k^2 & \text{in } [0,1] \end{cases}$$

with uniformly bounded coefficients $\lambda_{i,k}$, $i = 1, 2$, then u_k and v_k are uniformly bounded in the Lipschitz norm. In the recent paper [30], Soave and Zilio extended the result of [10] [14] [29] to the case of arbitrary number of components and general reaction terms. The approach here follows the mainstream of [30], based upon the blow-up technique and the almost monotonicity formula by Caffarelli-Jerison-Kenig.

The rest of the paper is organized as follows: Section 2 is devoted to giving some prior estimates. Section 3 deals with the blow-up analysis. In Section 4, we prove the uniform bound in the Lipschitz norm.

2. Some Preliminary Results

In this section, we will derive some basic estimates. As in [1] [4], if we let $\tilde{u}_k = e^{-\alpha m} u_k$, $\tilde{v}_k = e^{-\beta m} v_k$ then system (2) is equivalent to

$$\begin{cases} -\Delta \tilde{u}_k = \alpha \nabla \tilde{u}_k \cdot \nabla m + \left(m - b e^{\alpha m} \tilde{u}_k\right) \tilde{u}_k - k e^{\beta m} \tilde{u}_k \tilde{v}_k & \text{in } \Omega, \\ -\Delta \tilde{v}_k = \beta \nabla \tilde{v}_k \cdot \nabla m + \left(m - c e^{\beta m} \tilde{v}_k\right) \tilde{v}_k - k e^{\alpha m} \tilde{u}_k \tilde{v}_k & \text{in } \Omega, \\ \dfrac{\partial \tilde{u}_k}{\partial \nu} = \dfrac{\partial \tilde{v}_k}{\partial \nu} = 0 & \text{on } \partial\Omega. \end{cases} \tag{3}$$

We start with the following observation of system (3).

Lemma 2. *Let* $\tilde{u}_k = e^{-\alpha m} u_k$, $\tilde{v}_k = e^{-\beta m} v_k$ *and suppose that* (u_k, v_k) *is a non-negative solution of* (2). *Then for all* $x \in \bar{\Omega}$,

$$0 \le \tilde{u}_k(x) \le \max_{\bar{\Omega}} \left\{ m e^{-\alpha m} \right\} / b, \quad 0 \le \tilde{v}_k(x) \le \max_{\bar{\Omega}} \left\{ m e^{-\beta m} \right\} / c.$$

Furthermore,

$$0 \le u_k(x) \le \max_{\bar{\Omega}} \left\{ \frac{m}{b} \right\} e^{\alpha\left(\max_{\bar{\Omega}} m - \min_{\bar{\Omega}} m\right)}, v_k(x) \le \max_{\bar{\Omega}} \left\{ \frac{m}{c} \right\} e^{\beta\left(\max_{\bar{\Omega}} m - \min_{\bar{\Omega}} m\right)}.$$

Proof. We prove the estimate for $\tilde{u}_k$ and u_k; that for $\tilde{v}_k$ and v_k follows similarly. Let $x_0 \in \bar{\Omega}$ denote a point where $\tilde{u}_k(x_0) = \max_{\bar{\Omega}} \tilde{u}_k$. Assume by contradiction that

$$\tilde{u}_k(x_0) > \max_{\bar{\Omega}} \left\{ m\mathrm{e}^{-\alpha m} \right\} / b.$$

Since $\frac{\partial \tilde{u}_k}{\partial \nu} = 0$, then by the Hopf lemma $x_0 \notin \partial\Omega$. Hence, we have $x_0 \in \Omega$, $\nabla \tilde{u}_k(x_0) = \vec{0}$ and $\Delta \tilde{u}_k(x_0) \le 0$. It then follows that

$$\begin{aligned} 0 &\le k\mathrm{e}^{\beta m(x_0)} \tilde{u}_k(x_0) \tilde{v}_k(x_0) - \Delta \tilde{u}_k(x_0) \\ &= \alpha \nabla \tilde{u}_k(x_0) \cdot \nabla m(x_0) + \left(m(x_0) - b\mathrm{e}^{\alpha m(x_0)} \tilde{u}_k(x_0) \right) \tilde{u}_k(x_0) \\ &\le \left(\max_{\bar{\Omega}} \left\{ m\mathrm{e}^{-\alpha m} \right\} / b - \tilde{u}_k(x_0) \right) b\mathrm{e}^{\alpha m(x_0)} \tilde{u}_k(x_0) < 0. \end{aligned}$$

which is a contradiction. Hence, for all $x \in \bar{\Omega}$,

$$\tilde{u}_k(x) \le \max_{\bar{\Omega}} \left\{ m\mathrm{e}^{-\alpha m} \right\} / b.$$

and for all $x \in \bar{\Omega}$,

$$u_k(x) = \mathrm{e}^{\alpha m(x)} \tilde{u}_k(x) \le \left(\max_{\bar{\Omega}} m / b \right) \mathrm{e}^{\alpha \left(\max_{\bar{\Omega}} m - \min_{\bar{\Omega}} m \right)}.$$

This completes the proof of Lemma 2. □

In the blow up procedure, we need the following lemma, which extends the result in [11], Lemma 4.4.

Lemma 3. *Let* $B_{2R} = \left\{ x \in \mathbb{R}^N : |x|^2 < 2R \right\}$ *be the open ball in* $\mathbb{R}^N$. *Assume that* $u \in H^1(B_{2R})$ *satisfying*

$$\begin{cases} -\Delta u \le \theta \nabla m \cdot \nabla u - Hu & \text{in } B_{2R}, \\ u \ge 0 & \text{in } B_{2R}, \\ u \le A & \text{on } \partial B_{2R}, \end{cases}$$

where $m \in C^1(\bar{B}_{2R})$ and θ, H are two positive constant. Then for every $\delta \in (0,1)$,

$$\|u\|_{L^\infty(B_R)} \le CA\mathrm{e}^{-\delta R\sqrt{H}},$$

where C is a positive constant depending only on δ, R and $\|m\|_{C^1(\bar{B}_{2R})}$.

Proof. The proof is inspired by Conti *et al.* [11]. Let $\sigma = \|m\|_{C^1(\bar{B}_{2R})}$ and consider the following problem:

$$\begin{cases} \varphi''(r) + \frac{N-1}{r} \varphi'(r) = -\theta\sigma\varphi' + H\varphi, \\ \varphi(0) = \lambda > 0, \\ \varphi'(0) = 0. \end{cases} \tag{4}$$

We claim that:

1) $\varphi'(r) > 0$ for $r \in (0,\infty)$;
2) $\varphi(r) \le \lambda \mathrm{e}^{r\sqrt{H}}$ for $r \in [0,\infty)$;
3) $\varphi(r) \ge \frac{\lambda \mathrm{e}^{\theta\sigma(r_0 - r)}}{2} \left(\frac{r_0}{r} \right)^{N-1} \mathrm{e}^{(r - r_0)\sqrt{H}}$ for $r \in [r_0,\infty)$, where $r_0 > 0$.

To prove (1), we observe that φ is defined on $[0,\infty)$ and that $\varphi>0$, $\varphi'>0$ on $(0,\infty)$. Indeed, if not, φ is positive on $[0,R)$ and $\varphi(R)=0$, then $\varphi'(R)\le 0$; On the other hand, since

$$\left(\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'\right)'=\theta\sigma\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+\frac{N-1}{r}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi''=\mathrm{e}^{\theta\sigma r}r^{N-1}H\varphi,$$

then $\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'$ is strictly increasing on $[0,R]$. Hence, $\varphi'(R)>\varphi'(0)=0$, a contradiction. Since φ' is positive, we have $\varphi''\le H\varphi$. Then using the initial conditions and comparison arguments, $\varphi(r)\le\lambda\mathrm{e}^{r\sqrt{H}}$ for $r\in[0,\infty)$, and thus (2) follows. Finally, we define $\bar{\varphi}(r)=\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi(r)$. Then $\bar{\varphi}(r_0)\ge\lambda\mathrm{e}^{\theta\sigma r_0}r_0^{N-1}$ and

$$\bar{\varphi}'(r_0)=\theta\sigma\mathrm{e}^{\theta\sigma r_0}r_0^{N-1}\varphi(r_0)+\frac{N-1}{r_0}\mathrm{e}^{\theta\sigma r_0}r_0^{N-1}\varphi(r_0)+\mathrm{e}^{\theta\sigma r}r_0^{N-1}\varphi'(r_0)\ge 0.$$

Furthermore,

$$\begin{aligned}\bar{\varphi}''&=2\theta\sigma\frac{N-1}{r}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi+2\theta\sigma\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+(\theta\sigma)^2\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi\\&\quad+\frac{(N-1)(N-2)}{r^2}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi+2\frac{N-1}{r}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi''\\&=2\theta\sigma\frac{N-1}{r}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi+\theta\sigma\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+(\theta\sigma)^2\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi\\&\quad+\frac{(N-1)(N-2)}{r^2}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi+\frac{N-1}{r}\mathrm{e}^{\theta\sigma r}r^{N-1}\varphi'+H\bar{\varphi}\\&\ge H\bar{\varphi},\end{aligned}$$

since $\varphi>0$, $\varphi'>0$. Using again comparison arguments, we obtain

$$\bar{\varphi}(r)\ge\frac{\lambda\mathrm{e}^{\theta\sigma r_0}r_0^{N-1}}{2}\left(\mathrm{e}^{(r-r_0)\sqrt{H}}+\mathrm{e}^{-(r-r_0)\sqrt{H}}\right),$$

which gives (3).

Now let ψ be the solution of

$$\begin{cases}\psi''(r)+\dfrac{N-1}{r}\psi'(r)=-\theta\sigma\psi'+H\psi,\\ \psi(2R)=A,\\ \psi'(0)=0.\end{cases}$$

Clearly ψ satisfies the assumptions in (4) for a suitable λ, so $\psi'>0$. Recall that $\sigma=\|m\|_{C^1(B_{2R})}$, thus we have

$$\psi''(r)+\frac{N-1}{r}\psi'(r)\le-\theta\psi'\nabla m\cdot\frac{|x|}{r}+H\psi.$$

If we let $v(x)=\psi(|x|)$, then by construction we have that v is a radially symmetric function with $-\Delta v\ge\theta\nabla m\cdot\nabla v-Hv$ in B_{2R}, $v=A$ on ∂B_{2R}, and hence, by maximum principle, $0\le u(x)\le v(x)$ in B_{2R}. Moreover, since ψ is an increasing function, if we prove that $\psi(R)\le C(\delta,R,\sigma)A\mathrm{e}^{-\delta\sqrt{H}}$, then we will obtain the required bound for $\|u\|_{L^\infty(B_R)}$ and the proof of the lemma will be concluded. Using (3) and choosing $r_0=\tau R$, $\tau\in(0,1)$, we obtain

$$A=\psi(2R)\geq\frac{\lambda e^{\theta\sigma R(\tau-2)}}{2^N}\tau^{N-1}e^{(2-\tau)R\sqrt{H}},$$

that gives

$$\lambda\leq A\frac{2^N}{e^{\theta\sigma R(\tau-2)}\tau^{N-1}}e^{(-2+\tau)R\sqrt{H}}.$$

Substituting in the inequality in (2), we finally have

$$\psi(R)\leq A\frac{2^N}{e^{\theta\sigma R(\tau-2)}\tau^{N-1}}e^{(-1+\tau)R\sqrt{H}},$$

then setting $\delta=1-\tau$, provides the desired inequality. □

3. Asymptotic of the Blow up Sequence

We deduce from Section 2 that the solutions of system (2) is uniform bounded in $L^\infty(\Omega)$. For any compact set $K\subset K'\subset\subset\Omega$, we are aim to show that the Lipschitz semi-norm of solutions to system (2) is bounded in K, uniformly in k. To begin with, let η be a cut-off function such that $0\leq\eta\leq1$, $\eta\equiv1$ in K and supp $\eta=:K'\subset\subset\Omega$, we want to show that there exist a constant $C>0$ independent of k such that,

$$\max\left\{\sup_{x\in\Omega}\left|\nabla(\eta u_k)\right|,\sup_{x\in\Omega}\left|\nabla(\eta v_k)\right|\right\}\leq C,\tag{5}$$

from which the desired result follows. Inspired from the work of Soave and Zilio in [30], we assume by contradiction that, up to a subsequence, it holds

$$L_k:=\max\left\{\sup_{x\in\Omega}\left|\nabla(\eta u_k)\right|,\sup_{x\in\Omega}\left|\nabla(\eta v_k)\right|\right\}\to+\infty.$$

Without loss of generality, we may assume that the supremum is achieved by u_k at a point $x_k\in K'$, that is

$$L_k=\left|\nabla u_k(x_k)\right|\to+\infty.$$

Now we introduce two blow-up sequences

$$w_k(x)=\frac{\eta(x_k)u_k(x_k+r_kx)}{L_kr_k},\quad z_k(x)=\frac{\eta(x_k)v_k(x_k+r_kx)}{L_kr_k},\quad x\in\Omega_k,$$

$$\overline{w}_k(x)=\frac{(\eta u_k)(x_k+r_kx)}{L_kr_k},\quad \overline{z}_k(x)=\frac{(\eta v_k)(x_k+r_kx)}{L_kr_k},\quad x\in\Omega_k.$$

where $\Omega_k:=\frac{\Omega-x_k}{r_k}$. We choose the scaling factor $r_k>0$ in such a way that

$$\overline{w}_k(0)+\overline{z}_k(0)=\frac{(\eta u_k)(x_k)+(\eta v_k)(x_k)}{L_kr_k}=1$$

$$\Rightarrow r_k=\frac{(\eta u_k)(x_k)+(\eta v_k)(x_k)}{L_k}\to0.$$

Note that, since $K'\subset\subset\Omega$, we have $\Omega_k\to\mathbb{R}^N$ as $k\to\infty$. Furthermore, if (u_k,v_k) is a solution to (2), then (w_k,z_k) satisfies

$$\begin{cases} -\nabla\cdot\left(\nabla w_k-\alpha w_k\nabla m_k\right)=r_k^2 w_k m_k-bM_k w_k^2-kM_k w_k z_k & \text{in } \Omega_k, \\ -\nabla\cdot\left(\nabla z_k-\beta z_k\nabla m_k\right)=r_k^2 z_k m_k-cM_k z_k^2-kM_k w_k z_k & \text{in } \Omega_k, \end{cases} \tag{6}$$

where

$$m_k(x)=m(x_k+r_k x) \text{ and } M_k:=L_k r_k^3/\eta(x_k).$$

The following lemma focuses on some preliminary properties of the blow up sequences.

Lemma 4. *In the previous blow-up setting, the following assertions hold:*

1) $\nabla m_k\to\vec{0}$, $\Delta m_k\to 0$, uniformly in Ω_k as $k\to+\infty$, in particular,

$$m_k(x)\to m_0 \text{ for some constant } m_0>0;$$

2) we have

$$r_k^2 w_k m_k-bM_k w_k^2\to 0,\ r_k^2 z_k m_k-cM_k z_k^2\to 0,$$

uniformly in all Ω_k as $k\to\infty$;

3) the sequence $\{\overline{w}_k\}$ and $\{\overline{z}_k\}$ have uniformly bounded Lip-seminorm:

$$\max\left\{\sup_{x\neq y}\frac{\left|\overline{w}_k(x)-\overline{w}_k(y)\right|}{x-y},\sup_{x\neq y}\frac{\left|\overline{z}_k(x)-\overline{z}_k(y)\right|}{x-y}\right\}\le 1;$$

furthermore $\left|\nabla\overline{w}_k(0)\right|=1$ and $\left|\nabla w_k(0)\right|\to 1$ as $k\to+\infty$;

4) there exist w,z, globally Lipschitz continuous in $\mathbb{R}^N$ with Lipschitz constant equal to 1, such that up to a subsequence:

$$w_k\to w, z_k\to z \text{ in } C_{loc}\left(\mathbb{R}^N\right),$$

$$\overline{w}_k\to w, \overline{z}_k\to z \text{ in } C_{loc}\left(\mathbb{R}^N\right);$$

5) there holds $w_k\to w, z_k\to z$ in $H^1_{loc}\left(\mathbb{R}^N\right)$ as $k\to\infty$, and for any $r>0$ there exist $C>0$, independent of *k*, such that

$$\int_{B_r(0)} kM_k w_k z_k\le C. \tag{7}$$

If $kM_k\to\infty$, then $w_k z_k\to 0$ as $k\to\infty$. Moreover the limit w, z satisfies

$$\begin{cases} -\Delta w=0 & \text{in } \{w>0\}, \\ -\Delta z=0 & \text{in } \{z>0\}, \\ wz=0 & \text{in } \mathbb{R}^N, \\ w,z\ge 0 & \text{in } \mathbb{R}^N. \end{cases} \tag{8}$$

Proof. 1) Since $m\in C^2\left(\overline{\Omega}\right)$, then for every $x\in\Omega_k$,

$$\left|\nabla m_k(x)\right|=r_k\left|\nabla m(x_k+r_k x)\right|\le r_k\|m\|_{C^1(\overline{\Omega})}\to 0 \quad \text{as } k\to+\infty,$$

$$\left|\Delta m_k(x)\right|=r_k^2\left|\Delta m(x_k+r_k x)\right|\le r_k^2\|m\|_{C^2(\overline{\Omega})}\to 0 \quad \text{as } k\to+\infty.$$

Note also that $m(x)$ is positive somewhere in Ω, thus there exists a positive constant m_0, such that $m_k\to m_0$.

2) By Lemma 2 and the definitions of w_k and η, we have

$$r_k^2 w_k m_k - bM_k w_k^2 = \frac{r_k \eta(x_k)(u_k m_k)(x_k + r_k x)}{L_k} - b\frac{r_k \eta(x_k) u_k^2(x_k + r_k x)}{L_k}$$
$$\le C\frac{r_k}{L_k} \to 0.$$

Similarly, we have $r_k^2 z_k m_k - cM_k z_k^2 \to 0$.

The uniform bound on the Lipschitz seminorm of $\bar{w}_k$, $\bar{z}_k$ and the fact that $|\nabla \bar{w}_k(0)| = 1$, are direct consequence of the definitions. Moreover

$$\nabla \bar{w}_k(0) = \frac{u_k(x_k)\nabla \eta(x_k)}{L_k} + \frac{\eta(x_k)\nabla u_k(x_k)}{L_k} = o(1) + \nabla w_k(0),$$

as $k \to \infty$, then (3) holds.

4) We will only prove the estimate of w_k and $\bar{w}_k$, that of z_k and $\bar{z}_k$ are similar. For any fixed $r > 0$, we may let k sufficient large such that $\overline{B_r(0)} \subset\subset K$. The sequence $\{\bar{w}_k\}$ has a uniformly Lipschitz seminorm in $B_r(0)$, and is uniformly bounded in 0. Hence by the Ascoli-Arzelà theorem, it is uniformly convergent (up to a subsequence) to some $w \in C\left(\overline{B_r(0)}\right)$ having Lipschitz-seminorm bounded by 1. To complete the proof, we shall show that $w_k - \bar{w}_k \to 0$ as $k \to +\infty$ in $C_{loc}(\mathbb{R}^N)$. To this aim, it is sufficient to observe that for any compact $K \subset \mathbb{R}^N$,

$$\sup_{x\in K}|w_k(x) - \bar{w}_k(x)| = \sup_{x\in K}\frac{u_k(x_k + r_k x)}{L_k r_k}|\eta(x_k) - \eta(x_k + r_k x)|$$
$$\le \sup_{x\in K}\frac{lC}{L_k}|x|,$$

where l denotes the Lipschitz constant of η, C is the uniformly boundedness of $\{u_k\}$. Since $L_k \to \infty$ and K is compact, the desired result follows.

5) To prove (7), it is sufficient to test the equation for w_k against a smooth cut-off function $0 \le \varphi \le 1$ such that $\varphi = 1$ in $B_r(0)$ and $\varphi = 0$ in $\mathbb{R}^N \setminus B_{2r}(0)$, we obtain:

$$\int_{B_r(0)} kM_k w_k z_k \le \int_{B_{2r}(0)} \left|\left(r_k^2 m_k w_k - bM_k w_k^2\right)\varphi + w_k \Delta\varphi\right| + \int_{B_{2r}(0)} |\alpha w_k \nabla m_k \cdot \nabla \varphi|.$$

By the uniform boundedness of $\{w_k\}$ in compact sets and the fact that $|\nabla m_k| \to 0$, there exists a constant $C > 0$ independent of k, such that

$$\int_{B_r(0)} kM_k w_k z_k \le C.$$

Testing the equation for w_k against $w_k \varphi^2$, we also deduce that

$$\frac{1}{4}\int_{B_r(0)} |\nabla w_k|^2 \le 4\int_{B_{2r}(0)} w_k^2 \left(\varphi^2 \alpha^2 |\nabla m_k|^2 + |\nabla \varphi|^2\right)$$
$$+ 2\int_{B_{2r}(0)} \alpha w_k^2 \varphi |\nabla m_k||\nabla \varphi| + \left(r_k^2 m_k w_k - bM_k w_k^2\right) w_k \varphi^2 \le C,$$

where C is a positive constant independent of k. This implies that, up to a subsequence,

$$w_k \rightharpoonup w \text{ weakly in } H^1(B_r), \ w_k \to w \text{ in } L^2(B_r).$$

To prove the strong convergence, we test the equation for w_k against $(w_k - w)$, and recalling that $w_k \to w$ uniformly in B_r, we deduce that as $k \to \infty$,

$$\begin{aligned}
&\left|\int_{B_r} \nabla w_k \cdot \nabla (w_k - w)\right| \\
&= \left|\int_{\partial B_r} \partial_\nu w_k (w_k - w)\right| + \left|\int_{B_r} (-\alpha \nabla m_k \nabla w_k - \alpha w_k \Delta m_k \right. \\
&\quad \left. + r_k^2 m_k w_k - bM_k w_k^2)(w_k - w) - kM_k w_k z_k (w_k - w)\right| \\
&\le \|w_k - w\|_{L^\infty(B_r)} \int_{\partial B_r} |\partial_\nu w_k| \\
&\quad + \|w_k - w\|_{L^\infty(B_r)} \int_{B_r} \left|-\alpha \nabla m_k \nabla w_k - \alpha w_k \Delta m_k + r_k^2 m_k w_k - bM_k w_k^2\right| \\
&\to 0.
\end{aligned}$$

From this we can pass from the weak convergence to the strong one.

To prove (8), we note that

$$\begin{aligned}
-\Delta(w_k - z_k) = {} & (-\alpha \nabla w_k + \beta \nabla z_k)\nabla m_k + (-\alpha w_k + \beta z_k)\Delta m_k \\
& + (r_k^2 m_k w_k - bM_k w_k^2) - (r_k^2 m_k z_k - cM_k z_k^2).
\end{aligned}$$

By strong H^1 convergence and (1), above equation can be passing to the limit. So up to a subsequence, we have in the distribute sense that

$$-\Delta(w - z) = 0.$$

Since $wz = 0$, we have $w - z = w$ in $\{w > 0\}$, and thus

$$-\Delta w = 0 \text{ in } \{w > 0\}.$$

Similarly, the result holds for *z*. This completes the proof of Lemma 4. □

Lemma 5. *The limit function* (w, z) *is not constant. In particular, w is neither trivial nor constant.*

Proof. We divide the proof according to properties of kM_k.

Case 1. (kM_k) is bounded. The equation for w_k can be rewrite as:

$$-\Delta w_k + \alpha \nabla w_k \cdot \nabla m_k = -\alpha w_k \Delta m_k + r_k^2 m_k w_k - bM_k w_k^2 - kM_k w_k z_k.$$

Since $\{w_k\}$ is uniformly bounded in any compact set of $\mathbb{R}^N$, by standard regularity theory for elliptic equations, we deduce that for every compact $K \subset \mathbb{R}^N$ there exist $C > 0$ independent of *k* such that $\|w_k\|_{C^{1,\alpha}(K)} \le C$. This implies that, up to a subsequence

$$w_k \to w \text{ in } C^{1,\alpha}_{loc}(\mathbb{R}^N), \text{ for any } 0 < \alpha < 1.$$

So that in particular $|\nabla w(0)| = \lim_{k\to\infty} |\nabla w_k(0)| = 1$, and (w, z) cannot be a vector of constant functions.

Case 2. $kM_k \to +\infty$. By Lemma 4 (5) we infer that $wz \equiv 0$ in $\mathbb{R}^N$, and the choice of r_k implies that $w_k(0) + z_k(0) = 1$, so there are only two possibilities: either $w(0) = 0$, or $w(0) = 1$.

Assume at first that $w(0) = 0$, then $z(0) = 1$, and by continuity of w, z it results that $w \equiv 0$ in an open neighbourhood of 0. Moreover, there exists $0 < r \ll 1$, such that

$$z_k(0) \ge \frac{7}{8} \text{ in } B_r(0),$$

for sufficient large *k*. Thanks to Lemma 4 (4), we have

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$$\left|z_k(x)-z_k(0)\right|\le\left|z_k(x)-\overline{z}_k(x)\right|+\left|\overline{z}_k(x)-\overline{z}_k(0)\right|\le o(1)+|x|\le o(1)+\frac{1}{2},$$

as $k\to\infty$, for every $x\in B_{r/2}(0)$. Thus, whenever k is sufficiently large, $z_k\ge\frac{1}{8}$ in $B_{r/2}(0)$. As a consequence, if we Let $\tilde{w}_k=\mathrm{e}^{-\alpha m_k}w_k$, then $\tilde{w}_k$ satisfies

$$\begin{aligned}-\Delta\tilde{w}_k&=\alpha\nabla\tilde{w}_k\cdot\nabla m_k+\left(r_k^2m_k-bM_k\mathrm{e}^{\alpha m_k}\tilde{w}_k\right)\tilde{w}_k-kM_k\tilde{w}_kz_k\\&\le\alpha\nabla\tilde{w}_k\cdot\nabla m_k+\left(r_k^2m_k-bM_k\mathrm{e}^{\alpha m_k}\tilde{w}_k-\frac{1}{8}kM_k\right)\tilde{w}_k\\&\le\alpha\nabla\tilde{w}_k\cdot\nabla m_k-\frac{1}{16}kM_k\tilde{w}_k.\end{aligned}$$

By Lemma 3,

$$\tilde{w}_k\le C'\mathrm{e}^{-C''\sqrt{kM_k}}\ \text{ in } B_{r/4}.$$

Hence for every $x\in B_{r/4}(0)$,

$$\left|-\Delta\tilde{w}_k-\alpha\nabla\tilde{w}_k\cdot\nabla m_k\right|\le C.$$

Note that $\left|\nabla m_k\right|\to0$. By standard regularity theory for elliptic equations, we have $\left\|\tilde{w}_k\right\|_{C^{1+\alpha}\left(B_{r/4}\right)}\le C$. Note also that $w_k=\mathrm{e}^{\alpha m_k}\tilde{w}_k$ and $\left\|m_k\right\|_{C^2\left(B_r(0)\right)}\to0$ (by Lemma 4), we then deduce that

$$\left\|w_k\right\|_{C^{1+\alpha}}\le C\ \text{ in } B_{r/4}(0).$$

This implies that up to a subsequence $w_k\to w$ in $C^1\left(B_{1/4}\right)$. In particular $\left|\nabla w(0)\right|=1$, in contradiction with the fact that $w\equiv0$ in a neighbourhood of 0. Thus, the case $w(0)=0$ is impossible, therefore $w(0)=1$. As a consequence the same argument described above provides $w_k(x)\ge\frac{1}{8}$ in $B_{r/2}(0)$. If we let $\tilde{z}_k=\mathrm{e}^{-\beta m_k}z_k$, then $\tilde{z}_k$ satisfies

$$\begin{cases}-\Delta\tilde{z}_k\le\alpha\nabla\tilde{z}_k\cdot\nabla m_k-\frac{1}{16}kM_k\tilde{z}_k & \text{in } B_{r/2},\\ \tilde{z}_k\ge0 & \text{in } B_{r/2},\\ \tilde{z}_k\le A & \text{in } B_{r/2}.\end{cases}$$

By Lemma 3 again,

$$\tilde{z}_k\le C'\mathrm{e}^{-C''\sqrt{kM_k}}\ \text{ in } B_{r/4}.$$

By the uniform boundedness of the sequence $\{w_k\}$ in $B_{1/4}$, we infer that,

$$\left|-\Delta\tilde{w}_k-\alpha\nabla\tilde{w}_k\cdot\nabla m_k\right|\le C.$$

And hence up to a subsequence $w_k\to w$ in $C^1\left(B_{1/4}\right)$. In particular, by Lemma 4 (3) we have

$$\left|\nabla w(0)\right|=\lim_k\left|\nabla w_k(0)\right|=1,$$

which completes the proof. □

Lemma 6. *There exist* $C>0$ *such that* $kM_k\ge C$.

Proof. Let us assume by contradiction that there exists a subsequence $k_nM_{k_n}\to0$. Reasoning as in the previous lemma, the limiting function (w,z)

satisfies

$$\begin{cases} -\Delta w = 0 & \text{in } \mathbb{R}^N, \\ -\Delta z = 0 & \text{in } \mathbb{R}^N, \end{cases}$$

and $w, z \geq 0$, thus thanks to the Liouville theorem, w, z are constant. This contradicts the fact that $|\nabla w(0)| = 1$. □

We conclude this section by summing up what we proved so far in the following statement.

Proposition 7. *Under the previous notations, we have*

1) Up to a subsequence

$$w_n \to w, z_n \to z \text{ in } C_{loc}\left(\mathbb{R}^N\right),$$

$$\overline{w}_n \to w, \overline{z}_n \to z \text{ in } C_{loc}\left(\mathbb{R}^N\right);$$

w, z is non-trivial and non-constant, and in particular $|\nabla w(0)| = 1$;

2) There exist $C > 0$ such that $kM_k \geq C$;

3) If (kM_k) is bounded, then

$$\begin{cases} -\Delta w = -M_\infty wz & \text{in } \mathbb{R}^N, \\ -\Delta z = -M_\infty wz & \text{in } \mathbb{R}^N, \\ wz \geq 0 & \text{in } \mathbb{R}^N, \end{cases}$$

where $kM_k \to M_\infty$ as $k \to \infty$;

4) If $kM_k \to +\infty$, then both w and z are subharmonic in $\mathbb{R}^N$, and

$$\begin{cases} -\Delta w = 0 & \text{in } \{w > 0\}, \\ -\Delta z = 0 & \text{in } \{z > 0\}, \\ wz = 0 & \text{in } \mathbb{R}^N, \\ w, z \geq 0 & \text{in } \mathbb{R}^N. \end{cases}$$

4. Uniform Lipschitz Bounds with Respect to *k*

This section is devoted to the study of the Lipschitz uniform continuty of the system (2). In Section 3, we have proved that the limit (w, z) is non-trivial and non-constant, and in particular $|\nabla w(0)| = 1$ (Proposition 7). In what follows, we will show that one of the components of (w, z) is identically zero and the other is a constant, which bring us to a contradiction.

For any given $u, v \in H^1_{loc}\left(\mathbb{R}^N\right)$ functions, we let

$$\Phi(r) := \frac{1}{r^4} \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} \mathrm{d}x \int_{B_r} \frac{|\nabla v|^2}{|x|^{n-2}} \mathrm{d}x.$$

we shall make use of the celebrated almost monotonicity formula of Callarelli-Jerison-Kening, which we recall here in its original formulation.

Theorem 8. (*Callarelli-Jerison-Kening almost monotonicity*). *Suppose u, v are non-negative, continuous functions on the unit ball* B_1. *Suppose that* $-\Delta u \leq 1$ *and* $-\Delta v \leq 1$ *in the sense of distributions and that* $u(x)v(x) = 0$ *for all* $x \in B_1$. *Then there exist a constant C depending only on dimension such that* *for every* $0 < r \le 1$:

$$\Phi(r) \le C\left(1 + \int_{B_r} \frac{|\nabla u|^2}{|x|^{n-2}} + \int_{B_r} \frac{|\nabla v|^2}{|x|^{n-2}}\right).$$

Moreover, if u and v satisfy the same assumptions also in the ball B_2, then there exist a dimensional constant $C > 0$ such that

$$\Phi(r) \le C\left(1 + \int_{B_2} u^2 + \int_{B_2} v^2\right)^2,\ 0 < r \le 1.$$

Now we consider the following systems

$$\begin{cases} -\nabla \cdot (\nabla w_k - \alpha w_k \nabla m_k) = r_k^2 w_k m_k - bM_k w_k^2 - kM_k w_k z_k & \text{in } \Omega_k, \\ -\nabla \cdot (\nabla z_k - \beta z_k \nabla m_k) = r_k^2 z_k m_k - cM_k z_k^2 - kM_k w_k z_k & \text{in } \Omega_k. \end{cases}$$

Therefore,

$$\begin{aligned} -\Delta(w_k - z_k) &= (-\alpha \nabla w_k + \beta \nabla z_k) \nabla m_k + (-\alpha w_k + \beta z_k) \Delta m_k \\ &\quad + (r_k^2 m_k w_k - bM_k w_k^2) - (r_k^2 m_k z_k - cM_k z_k^2). \end{aligned}$$

Notice that $r_k^2 m_k w_k - bM_k w_k^2 \to 0$, $r_k^2 m_k z_k - cM_k z_k^2 \to 0$, $m_k \to m_0$. Hence in the sense of distributions that $-\Delta(w_k - z_k) \to 0$, and in particular

$$\begin{cases} -\Delta(w_k - z_k)^+ \le 1, \\ -\Delta(w_k - z_k)^- \le 1. \end{cases} \tag{9}$$

for k sufficiently large.

Lemma 9. *There exist a constant* $C > 0$ *independent of k such that for any* $r \in (0, +\infty)$ *and* $x_0 \in \Omega_k$,

$$\frac{1}{r^N} \int_{B_r(x_0)} \left|\nabla (w_k - z_k)^+\right|^2 \cdot \frac{1}{r^N} \int_{B_r(x_0)} \left|\nabla (w_k - z_k)^-\right|^2 \le C. \tag{10}$$

Proof. By (9), it follows that the positive and negative part of $(w_k - z_k)$ fall under the assumptions of Theorem 8, and in particular

$$\begin{aligned} &\frac{1}{r^{2N}} \int_{B_r(x_0)} \left|\nabla (w_k - z_k)^+\right|^2 \int_{B_r(x_0)} \left|\nabla (w_k - z_k)^-\right|^2 \\ &\le \frac{1}{r^4} \int_{B_r(x_0)} \frac{\left|\nabla (w_k - z_k)^+\right|^2}{|x - x_0|^{N-2}} \cdot \int_{B_r(x_0)} \frac{\left|\nabla (w_k - z_k)^-\right|^2}{|x - x_0|^{N-2}} \\ &\le C\left(1 + \int_{B_r(x_0)} w_k^2 + z_k^2\right)^2 \le C, \end{aligned}$$

where $C > 0$ is independent of *k*. □

Corollary 1. *Any blow-up limit* (w, z) *is made of ordered functions, that is if*

$$w_k \to w,\ z_k \to z,$$

then either $w \le z$ or $z \le w$, in $C_{loc}(\mathbb{R}^N)$.

Proof. Indeed, scaling properly of the estimate (10), we obtain for every $r \in (0, 1/r_k)$ and *k* large enough

$$\frac{1}{r^N}\int_{B_r(x_0)}\left|\nabla\left(w_k-z_k\right)^+\right|^2\cdot\frac{1}{r^N}\int_{B_r(x_0)}\left|\nabla\left(w_k-z_k\right)^-\right|^2\le\frac{\eta\left(x_k\right)^4}{L_k^4}\to 0$$

as $k\to+\infty$. The conclusion follows by strong $H_{loc}^1\left(\mathbb{R}^N\right)$ convergence of the blow-up sequence and by the continuity of the blow-up limit. □

In order to complete the proof of Theorem 1, we need the following classical result, for which we refer to Lemma 2 in [31].

Lemma 10. *Let* $1<p<\infty$, *and let* $u\in L_{loc}^p\left(\mathbb{R}^N\right)$ *be a solution of*

$$-\Delta u\le-|u|^{p-1}u \text{ in } \mathbb{R}^N,$$

if we assume u to be non-negative, then $u\equiv 0$.

With the lemmas above, we can now complete the proof of uniform Lipschitz bounds.

Proof of Theorem 1. According to kM_k, we divided the proof in two steps.

Step 1. The case (kM_k) bounded. In this case by Proposition 7 the limiting function (w,z) is a non-negative, non-trivial, non-constant and sublinear solution of

$$-\Delta w=-M_\infty wz,\quad -\Delta z=-M_\infty wz.$$

By Corollary 1, we evince that either $w\le z$ in $\mathbb{R}^N$, or $w\ge z$ in $\mathbb{R}^N$. Without loss of generality, we suppose that $w\ne 0$ and $w\ge z$. Thus

$$-\Delta z=-M_\infty wz\le-M_\infty z^2.$$

Thanks to Lemma 10, we have $z\equiv 0$. But then

$$-\Delta w=-M_\infty wz\equiv 0.$$

Then by the classical Liouville theorem, we have w is a constant, which is in contradiction with the fact that w is non-trivial and non-constant.

Step 2. the case $kM_k\to+\infty$. In such a situation, $wz\equiv 0$. Notice that

$$\begin{aligned}-\Delta\left(w_k-z_k\right)=&\left(-\alpha\nabla w_k+\beta\nabla z_k\right)\nabla m_k+\left(-\alpha w_k+\beta z_k\right)\Delta m_k\\&+\left(r_k^2m_kw_k-bM_kw_k^2\right)-\left(r_k^2m_kz_k-cM_kz_k^2\right)\to 0.\end{aligned}$$

that is $-\Delta(w-z)=0$. Then Corollary 1 implies that either $w\le z$, or $w\ge z$. Without loss of generality, we suppose that $w\ge z$, then the classical Liouville theorem shows that

$$w-z\equiv C\ge 0.$$

since $w\cdot z=0$, Therefore $(z+C)z=0$.

We deduce that $z=0$, and $w\equiv C$, this implies that w is a constant, similarly, a contradiction. This completes the proof of Theorem 1. □

5. Conclusion and Further Works

The study of the asymptotic behavior of singular perturbed equations and systems of elliptic or parabolic type is very broad and active subject of research. In this paper, we study a competition-diffusion-advection system for two competing species in a spatially heterogenous environment. We prove the uniform Lipschitz bound for solutions of the system, which extends known quasi-optimal results and covers the optimal case for this problem. We remark that the existence of uniform Lipschitz bounds is relevant not only for a pure mathematical flavour. As already observed in [29], it is necessary to obtain, rigorous qualitative description of phase separation phenomena (the uniform Hölder bounds would not be sufficient for this purpose.)

Finally, we mention that there are many interesting problems for further study. Note that we established uniform Lipschitz bound for solutions to elliptic system (2), naturally to ask whether our results can be extended to the parabolic system (1)? Up to our knowledge, the optimal Lipschitz bound for parabolic setting is unknown even for the case when $\alpha = \beta = 0$ (without advection terms) in system (1). Moreover, in system (2) the advection rates α and β are fixed nonzero constants, what happens if α and β are *k*-dependent and are suitably large? In such situation, the regularity of the solutions remains a challenge, and it will be the object of a forthcoming paper.

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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