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Sign Changing Solution of a Semilinear Schrödinger Equation with Constraint

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Abstract

The purpose of this paper is to study a semilinear Schrödinger equation with constraint in $H^1(\mathbf{R}^N)$, and prove the existence of sign changing solution. Under suitable conditions, we obtain a negative solution, a positive solution and a sign changing solution by using variational methods.

Keywords

Semilinear Schrödinger Equation, Sign Changing Solution, Palais-Smale Condition, Pseudo-Gradient Vector Field

1. Introduction

This article deals with the following semilinear Schrödinger equation with constraint

$$\begin{cases} -\Delta u + V(x)u = \lambda f(x,u), \quad x \in \mathbf{R}^N,\ u \in H^1(\mathbf{R}^N), \\ \int_{\mathbf{R}^N} \left(|\nabla u|^2 + V(x)u^2\right) \mathrm{d}x = r^2, \\ u(x) \to 0, \quad |x| \to +\infty. \end{cases} \tag{1.1}$$

Given $r > 0$, we try to find (u,λ) to satisfy the Equation (1.1). We say (u,λ) a positive solution if u is positive, a negative solution if u is negative, and a sign changing solution if u is sign changing.

Several authors have considered a Schrödinger equation of the form

$$\begin{cases} -\Delta u + V(x)u = f(x,u), \quad x \in \mathbf{R}^N,\ u \in H^1(\mathbf{R}^N), \\ u(x) \to 0, \quad |x| \to +\infty. \end{cases} \tag{1.2}$$

In Bartsch and Wang [1], it is shown that the problem (1.2) possesses infinitely many solutions when $f(x,u)$ is odd with respect to *u*. Liu [2] obtains a positive solution and a negative solution of the problem (1.2) under the assumption that $V(x)$ and $f(x,u)$ are periodic with respect to the x-variables. Bartsch, Liu and Weth [3] prove the existence of sign changing solutions to the problem (1.2) and estimate the number of nodal domain.

Some papers concern with the problem (1.1). Under some conditions, a positive and a negative solution can be found in [4] and [5]. [6] gives some results on the existence of sign changing and multiple solutions of the problem (1.1) with different conditions.

In order to state our results, we require the following assumptions:

(A_1) $V \in L^\infty_{loc}(\mathbf{R}^N)$, $\inf_{x\in\mathbf{R}^N} V(x) > 0$.

(A_2) $f:\mathbf{R}^N \times \mathbf{R} \to \mathbf{R}$ is locally Lipschitz continuous, and there are constants $C > 0$ and $p \in (2, 2^*)$ such that

$$|f(x,t)| \le C\left(1+|t|^{p-1}\right) \text{ for } x \in \mathbf{R}^N, t \in \mathbf{R},$$

where $2^* := \frac{2N}{N-2}$ for $N \ge 3$ and $2^* := \infty$ for $N = 1,2$. Moreover, $f(x,t) = o(|t|)$ as $t \to 0$ uniformly in *x*.

(A_3) There is a constant $\eta > 2$ such that

$$0 \le \eta F(x,t) \le f(x,t)t \text{ for } x \in \mathbf{R}^N,\ t \in \mathbf{R},$$

where $F(x,t) := \int_0^t f(x,s)\mathrm{d}s$ for $x \in \mathbf{R}^N, t \in \mathbf{R}$.

(A_4) There is an open subset $\Omega \subset \mathbf{R}^N$ such that $tf(x,t) > 0$ for $x \in \Omega$ and $|t|$ sufficiently large.

(A_5) $\lim_{|x|\to+\infty} \sup_{|t|\le r} \frac{|f(x,t)|}{|t|} = 0$ for every $r > 0$.

Our main result is the following theorem.

Theorem 1.1 Suppose (A_1)-(A_5) hold. Then problem (1.1) has at least three nontrivial solutions u_+, u_-, and $\bar{u}$, where u_+ is positive, and u_- is negative and $\bar{u}$ changes sign.

The key point is to construct certain invariant sets of the gradient flow associated with the energy functional of the elliptic problem. All positive and negative solutions are contained in these invariant sets. And minimax procedures can be used to construct sign changing critical point of the energy functional outside these invariant sets.

2. Preliminaries

We first fix some notations. Denote the usual Sobolev space by $W^{m,p}(\mathbf{R}^N)$, and set $H^m(\mathbf{R}^N) = W^{m,2}(\mathbf{R}^N)$. Consider the Hilbert space

$$H := \left\{u \in H^1(\mathbf{R}^N) : \int_{\mathbf{R}^N} V(x)u^2 \mathrm{d}x < +\infty\right\}.$$

We introduce the inner product in H by the formula

$$\langle u,v\rangle := \int_{\mathbf{R}^N} \left(\nabla u \cdot \nabla v + V(x)uv\right)\mathrm{d}x \text{ for } u,v \in H,$$

and the corresponding norm

$$\|u\| := \sqrt{\langle u,u\rangle} \ \text{ for } u \in H.$$

According to (A_1), there is a continuous embedding $H \hookrightarrow H^1(\mathbf{R}^N)$, hence

$$H \hookrightarrow L^s(\mathbf{R}^N) \ \text{ for } 2 \le s \le 2^*. \tag{2.1}$$

Note that by (A_2) for any $\varepsilon > 0$, there is a constant $K_1(\varepsilon) > 0$ such that

$$|f(x,t)| \le \varepsilon|t| + K_1(\varepsilon)|t|^{p-1} \ \text{ for } x \in \mathbf{R}^N, t \in \mathbf{R}. \tag{2.2}$$

Assumption (A_3) implies that given $\delta > 0$, there exists a constant $K_2(\delta) > 0$ such that

$$F(x,t) \ge K_2(\delta)|t|^\eta - \delta|t|^2 \ \text{ for } x \in \mathbf{R}^N, t \in \mathbf{R}. \tag{2.3}$$

Denote

$$S_r := \left\{u \in H : \int_{\mathbf{R}^N}\left(|\nabla u|^2 + V(x)u^2\right)\mathrm{d}x = r^2\right\},$$

$$F(x,u) := \int_0^u f(x,t)\,\mathrm{d}t \ \text{ for } u \in \mathbf{R}, \tag{2.4}$$

$$J(u) := -\int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x \ \text{ for } u \in H, \tag{2.5}$$

$$I := J|_{S_r}. \tag{2.6}$$

By Zeidler [7], we have

$$I'(u) = J'(u) - \frac{\langle J'(u),u\rangle}{\|u\|^2}u \ \text{ for } u \in S_r, \tag{2.7}$$

where $\langle J'(u),v\rangle = -\int_{\mathbf{R}^N} f(x,u)v\,\mathrm{d}x$ for $u,v \in H$. It is easy to see from (2.7) that the critical points of I correspond to the solutions of problem (1.1) with $\lambda = -\frac{\|u\|^2}{\langle J'(u),u\rangle}$. And I is bounded.

Definition 2.1 Suppose E is a real Banach space. For $\Phi \in \mathbf{C}^1(E,\mathbf{R})$, we say Φ satisfies the Palais-Smale condition (denoted by (PS)) if any sequence $\{u_n\} \subset E$ for which $\{\Phi(u_n)\}$ is bounded and $\Phi'(u_n) \to 0$ possesses a convergent subsequence. We say Φ satisfies $(PS)_c$ for a fixed $c \in \mathbf{R}$ if any sequence $\{u_n\} \subset E$ for which $\Phi(u_n) \to c$ and $\Phi'(u_n) \to 0$ possesses a convergent subsequence. We say Φ satisfies $(PS)^+$ if Φ satisfies $(PS)_c$ for all $c > 0$; Φ satisfies $(PS)^-$ if I satisfies $(PS)_c$ for all $c < 0$.

Lemma 2.1 [8] I satisfies $(PC)^-$.

Let **G** be the Nemytskii operator induced by f, the mapping I' may be written as

$$I'(u) = -T(u)u - \mathbf{KG}(u) \ \text{ for } u \in S_r, \tag{2.8}$$

where $T(u) := -\frac{1}{r^2}\int_{\mathbf{R}^N} f(x,u)u\,\mathrm{d}x$ and $\mathbf{K} := (-\Delta + V)^{-1}$. Note that

$$\mathbf{KG}(u) = (-\Delta + V)^{-1} f(\cdot,u(\cdot)) \ \text{ for } u \in H.$$

In other words, **KG** is uniquely determined by the relation

$$\langle \mathbf{KG}(u), v\rangle = \int_{\mathbf{R}^N} f(x,u)v\mathrm{d}x, \quad \text{for } u, v \in H. \tag{2.9}$$

$I'(u)$ is globally Lipschitz continuous in H applying (A_2) [6].

Let E be a real Banach space, $\Phi \in \mathbf{C}^1(E,\mathbf{R})$ and $\bar{E} := \{u \in E : \Phi'(u) \neq 0\}$. We will give some relevant definitions below.

Definition 2.2 A locally Lipschitz continuous mapping $Q : \bar{E} \to E$ is called a pseudo-gradient vector field (denoted by p.g.v.f) for Φ on $\bar{E}$ if it satisfies the following conditions

1) $\|Q(u)\|_E \le 2\|\Phi'(u)\|_{E^*}$;

2) $\langle \Phi'(u), Q(u)\rangle_E \ge \|\Phi'(u)\|_{E^*}^2$.

Suppose Q is a p.g.v.f for Φ on $\bar{E}$, and consider the initial value problem in $\bar{E}$

$$\begin{cases} \dfrac{\mathrm{d}}{\mathrm{d}t}\sigma(t,u) = -Q(\sigma(t,u)), & t \ge 0 \\ \sigma(0,u) = u. \end{cases} \tag{2.10}$$

According to the theory of ordinary differential equations in Banach space [9], (2.10) has a unique solution in $\bar{E}$, denoted by $\sigma(t,u)$, with right maximal interval of existence $[0, \omega_+(u))$. Note that $\omega_+(u)$ may be either a positive number or $+\infty$. Note also that $\Phi(\sigma(t,u))$ is monotonically decreasing on $[0, \omega_+(u))$ and therefore $\sigma(t,u)(0 \le t < \omega_+(u))$ is called a descending flow curve.

Definition 2.3 A nonempty subset M of E is called an invariant set of descending flow for Φ determined by Q if

$$\{\sigma(t,u) \mid 0 \le t < \omega_+(u)\} \subset M$$

for all $u \in \bar{E} \cap M$.

Definition 2.4 Let M and D be invariant sets of descending flow for $\Phi, D \subset M$. Denote

$$C_M(D) = \{u \mid u \in D, \text{or } u \in M \setminus D \text{ and there is } 0 \le t < \omega_+(u) \text{ such that } \sigma(t,u) \in D\}$$

If $D = C_M(D)$, then D is called a complete invariant set of descending flow relative to M.

3. Invariant Subsets of the Descending Flow

In this section we shall recall some results about the flow generated by I'. We refer to Mawhin and Willem [10] for details.

It is clear that I' is globally Lipschitz continuous, and I' is a p.g.v.f of I. In the following we consider the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = -I'(\sigma) = T(\sigma)\sigma + \mathbf{KG}(\sigma), & t \ge 0, \\ \sigma(0,u) = u, & u \in S_r. \end{cases} \tag{3.1}$$

Applying the theory of ordinary differential equations, we obtain:

Lemma 3.1 [10] There exists a unique solution $t \mapsto \sigma(t,u)$ of (3.1) defined on a maximal interval $[0,\omega_+(u))$ with $0<\omega_+(u)\le +\infty$. The flow $\sigma : D \mapsto S_r$ is continuous, where $D := \{(t,u): 0 \le t < \omega_+(u), u \in S_r\}$. For $0 \le t < \omega_+(u)$, σ has the expression

$$\sigma(t,u) = \mathrm{e}^{\int_0^t T(\sigma(s,u))\mathrm{d}s}\left(u + \int_0^t \mathrm{e}^{-\int_0^s T(\sigma(\xi,u))\mathrm{d}\xi}\mathbf{KG}(\sigma(s,u))\mathrm{d}s\right). \tag{3.2}$$

Lemma 3.2 [10] If $\omega_+(u)$ is finite, then $I(\sigma(t,u)) \to -\infty$ as $t \to \omega_+(u)$.

In our case, I is bounded and so it follows from Lemma 3.2 that $\omega_+(u) = +\infty$ for $u \in S_r$.

Lemma 3.3 [10] Suppose $c<b<0$, for any $u \in I^{-1}([c,b])$, either there exists a unique $t(u) \in [0,+\infty)$ such that $I(\sigma(t(u),u)) = c$ or there is a critical point v of I in $I^{-1}([c,b])$, such that $\sigma(t,u) \to v$ as $t \to +\infty$.

It is easy to verify that $\frac{\mathrm{d}\left(\|\sigma\|^2\right)}{\mathrm{d}t} = 0$, that is

$$\sigma(t,u) \in S_r \ \text{ for } u \in S_r. \tag{3.3}$$

In our further proof, we shall need the following Lemma which is derived by Brézis and extended by Martin to infinite dimensional space (cf. Theorem 1.6.3 in [11]).

Lemma 3.4 [11] Suppose E is a real Banach space, D is a closed subset of E, $Q: E \to E$ is locally Lipschitz continuous and

$$\lim_{h \downarrow 0} \frac{dist_E(u + hQ(u), D)}{h} = 0 \ \text{ for } u \in \partial D. \tag{3.4}$$

where $dist_E(.,.)$ is the distance on E. If $u_0 \in D$ and $\sigma(t)$ with $0 \le t < \omega_+(u_0)$ is the solution of the initial value problem

$$\begin{cases} \dfrac{\mathrm{d}\sigma}{\mathrm{d}t} = Q(\sigma), \\ \sigma(0,u_0) = u_0, \end{cases}$$

then $\sigma(t) \in D$ for all $t \in [0,\omega_+(u_0))$.

Next we will discuss the convex cones $P^+ := \{u \in H : u \ge 0\}$, and $P^- := \{u \in H : u \le 0\}$. Moreover, for $u \in H$ we denote that $u^+ := \max\{u,0\}$ and $u^- := \min\{u,0\}$. Note that $u \in H$ implies $u^\pm \in P^\pm$. Consider the sets

$$P_\varepsilon^+ := \{u \in H : dist(u,P^+) < \varepsilon\}, \quad P_\varepsilon^- := \{u \in H : dist(u,P^-) < \varepsilon\},$$

as well as $P_\varepsilon := \overline{P_\varepsilon^+} \cup \overline{P_\varepsilon^-}$ for $\varepsilon > 0$. Note that P_ε^+ and P_ε^- are open convex subsets of H, whereas P_ε is a closed and symmetric subset of H. Moreover, $H \setminus P_\varepsilon$ contains only sign changing functions.

Note that I' is a p.g.v.f for I, we can obtain a flow $\sigma : \mathscr{D} \to E$ satisfying (3.1) for all $(t,u) \in \mathscr{D} := \{(t,u): u \in \bar{H}, 0 \le t < \omega_+(u)\}$, where $\omega_+(u) \in (0,+\infty]$ is the maximal existence time for the trajectory $\sigma(t,u)$. We call σ the descending flow associated with I'. A subset $M \subset H$ is invariant for the σ if

$$\sigma(t,u) \in M, \ \text{for every } u \in M \text{ and every } t \in [0,\omega_+(u)).$$

If M is an invariant subset of H, we also consider

$$\mathscr{A}(M) := \{u \in H : \sigma(t,u) \in M \text{ for some } t \in (0, \omega_+(u))\},$$

and in addition we put

$$\mathscr{A}_0 := \{u \in H : \sigma(t,u) \to 0 \text{ as } t \to \omega_+(u)\}.$$

Note that $\mathscr{A}_0$ is open.

4. Three Solutions with One Changing Sign

In this section, we will give some proposition for finding three solutions with one changing sign.

Proposition 4.1 Suppose W is a finite dimensional subspace of H, there holds:

1) $\sup J(W) < +\infty$;

2) If $\sup J(C) < +\infty$, where $C := \{tu : t \geq 0, u \in S\}$ and S is a closed subset of some finite dimensional subspace W of H, then there is a constant $R > 0$ such that

$$J(u) < -\frac{1}{2}\|u\|^2 \text{ for } u \in C \setminus B_R(0). \tag{4.1}$$

Proof. 1) Obviously.

2) If we define

$$\Phi(u) := \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x, u(x))\mathrm{d}x \text{ for } x \in \mathbf{R}^N, u \in H,$$

then

$$J(u) = \Phi(u) - \frac{1}{2}\|u\|^2 \text{ for } u \in H.$$

Inequality (2.3) implies that for any $\delta > 0$ there exist constants $a_1, a_2 > 0$, such that

$$\Phi(u) \leq \frac{1}{2}\|u\|^2 - K(\delta)\int_{\mathbf{R}^N}|u|^\eta \mathrm{d}x + \delta\int_{\mathbf{R}^N}|u|^2 \mathrm{d}x \leq a_1\|u\|^2 - a_2\|u\|^\eta.$$

Hence for $R = (a_1 a_2)^{\frac{1}{\eta-2}}$ we have

$$\Phi(u) < 0, \text{ for } u \in C \setminus B_R(0).$$

Thus (4.1) hold. □

Using (2.7), we can note that

$$I'(u) = u - \frac{\mathbf{KG}(u)}{T(u)} = u - A(u) \text{ for } u \in S_r, \tag{4.2}$$

where $A(u) := \dfrac{\mathbf{KG}(u)}{T(u)}$.

Proposition 4.2 There exists $\varepsilon_0 > 0$ such that for $0 < \varepsilon \leq \varepsilon_0$, there holds

1) If $u \in P_\varepsilon^\pm \cap S_r$ and $t \in [0, \omega_+(u))$, then $\sigma(t,u) \in int(P_\varepsilon^\pm) \cap S_r$;

2) Every nontrivial solution $u \in P_\varepsilon^- \cap S_r$ of (1.1) is negative, and every non-

trivial solution $u \in P_\varepsilon^+ \cap S_r$ of (1.1) is positive.

Proof. 1) Let $d := \frac{1}{2}\inf_{x\in\mathbf{R}^N} V(x) > 0$, $u \in S_r$ and $v = \mathbf{KG}(u)$, then

$$\|u^+\|_{L^2} = \min_{y\in P^-}\|u-y\|_{L^2} \le \frac{1}{\sqrt{2d}}\min_{y\in P^-}\|u-y\| = \frac{1}{\sqrt{2d}} dist(u, P^-).$$

Similarly, using (2.1) we find for every $s \in [2, 2^*]$, there is a constant $C_s > 0$ with

$$\|u^\pm\|_{L^s} \le C_s dist(u, P^\mp). \tag{4.3}$$

Since $v^- \in P^-, v - v^- = v^+ \in P^+$, we have

$$dist(v, P^-) \le \|v - v^-\| = \|v^+\|.$$

It follows from (2.2), (2.9) and (4.3) that

$$\begin{aligned} dist(v, P^-)\|v^+\| &\le \|v^+\|^2 = \langle v, v^+\rangle = \langle \mathbf{KG}(u), v^+\rangle = \int_{\mathbf{R}^N} f(x,u)v^+ \mathrm{d}x \\ &\le \int_{\mathbf{R}^N} f(x,u)^+ v^+ \mathrm{d}x = \int_{\mathbf{R}^N} f(x,u^+)v^+ \mathrm{d}x \\ &\le \int_{\mathbf{R}^N}\left(d|u^+| + K(d)|u^+|^{p-1}\right)v^+ \mathrm{d}x \\ &\le d\|u^+\|_{L^2}\|v^+\|_{L^2} + K(d)\|u^+\|_{L^p}^{p-1}\|v^+\|_{L^p} \\ &\le \left(\frac{1}{2}dist(u, P^-) + \bar{K}dist(u, P^-)^{p-1}\right)\|v^+\|, \end{aligned}$$

with a constant $\bar{K} > 0$. Hence

$$dist(v, P^-) \le \frac{1}{2}dist(u, P^-) + \bar{K}dist(u, P^-)^{p-1}.$$

So there exists $\varepsilon_0 > 0$ such that

$$dist(\mathbf{KG}(u), P^-) \le \frac{3}{4}dist(u, P^-)$$

for every $u \in P_\varepsilon^- \cap S_r$ with $0 < \varepsilon \le \varepsilon_0$. Thus

$$\mathbf{KG}(u) \in int(P_\varepsilon^-) \text{ for } u \in P_\varepsilon^- \cap S_r. \tag{4.4}$$

For any $u \in P_\varepsilon^-$, we can choose $\delta > 0$ small enough such that

$$u + h(T(u)u + \mathbf{KG}(u)) = (1 + hT(u))u + h\mathbf{KG}(u) \in P_\varepsilon^- \quad \text{for } h \in (0, \delta)$$

Thus

$$\lim_{h\downarrow 0}\frac{dist(u - hI'(u), P_\varepsilon^-)}{h} = 0 \text{ for } u \in \partial P_\varepsilon^-. \tag{4.5}$$

It follows from Lemma 3.4 that if $\sigma(t,u)$ is the solution of (3.1), then it will hold that $\sigma(t,u) \in P_\varepsilon^-$ for all $t \in [0, \omega_+(u))$. So we can obtain from (4.4) that

$$\mathbf{KG}(\sigma(t,u)) \in int(P_\varepsilon^-) \quad \text{for } u \in P_\varepsilon^- \cap S_r, t > 0. \tag{4.6}$$

Set $\alpha(t) := -\int_0^t T(\sigma(s,u))\mathrm{d}s$, then $\alpha'(t) > 0, \alpha(t) > 0$, and $\alpha(t)$ is strictly increasing. Applying (4.6), we have

$$\frac{\mathbf{KG}(\sigma(t,u))}{\alpha'(t)} \in int(P_\varepsilon^-) \text{ for } u \in P_\varepsilon^- \cap S_r. \tag{4.7}$$

If we define $B(t) := \frac{\mathbf{KG}(\sigma(t,u))}{\alpha'(t)}$, and $F_t := \{B(s): 0 \le s \le t\}$, then F_t is a compact set of *H*. According to (4.7), $F_t \subset int(P_\varepsilon^-)$ and hence $\overline{co}F_t \subset int(P_\varepsilon^-)$, where $\overline{co}F_t$ is the closed convex hull of F_t in *H*. Note that

$$\begin{aligned}
&\frac{1}{e^{\alpha(t)}-1}\int_0^t e^{\alpha(s)}\mathbf{KG}(\sigma(s,u))ds \\
&= \frac{1}{e^{\alpha(t)}-1}\int_1^{e^{\alpha(t)}} \frac{\mathbf{KG}(\sigma(\alpha^{-1}(\ln s),u))}{\alpha'(\alpha^{-1}(\ln s))}ds \\
&= \lim_{m\to+\infty}\frac{1}{m}\sum_{i=1}^m A\left(\alpha^{-1}\left(\ln\left(1+\frac{i}{m}\left(e^{\alpha(t)}-1\right)\right)\right)\right) \in \overline{co}F_t \subset int(P_\varepsilon^-).
\end{aligned}$$

From (3.2) we get

$$\sigma(t,u) = e^{-\alpha(t)}u + \frac{1-e^{-\alpha(t)}}{e^{\alpha(t)}-1}\int_0^t e^{\alpha(s)}\mathbf{KG}(\sigma(s,u))ds. \tag{4.8}$$

Denote $F := \left\{u, \frac{1}{e^{\alpha(t)}-1}\int_0^t e^{\alpha(s)}\mathbf{KG}(\sigma(s,u))ds\right\}$, then *F* is also a compact set of *H*. Using by (4.6) and (4.8), we obtain that

$$\sigma(t,u) \in \overline{co}F \subset int(P_\varepsilon^-).$$

Hence $\sigma(t,u) \in int(P_\varepsilon^-) \cap S_r$ for $u \in P_\varepsilon^- \cap S_r$ and $t \in [0, \omega_+(u))$. And $\sigma(t,u) \in int(P_\varepsilon^+) \cap S_r$ for $u \in P_\varepsilon^+ \cap S_r$ and $t \in [0, \omega_+(u))$ can be proved analogously.

(2) Put $w = -\frac{\mathbf{KG}(u)}{T(u)}$, it follows from (2.9) that

$$\begin{aligned}
dist(w, P^-)\|w^+\| \le \|w^+\|^2 = \langle w, w^+\rangle &= \left\langle -\frac{\mathbf{KG}(u)}{T(u)}, w^+\right\rangle \\
&= r^2\left(\int_{\mathbf{R}^N} f(x,u)u dx\right)^{-1}\int_{\mathbf{R}^N} f(x,u)w^+ dx.
\end{aligned}$$

Any $u \in S_r$, $\frac{1}{r^2}\int_{\mathbf{R}^N} f(x,u)u dx > 0$. By (2.2) and (4.3), for $0 < \xi < \frac{d}{r^2}\int_{\mathbf{R}^N} f(x,u)u dx < K_1$ with a constant $K_1 > 0$, we get

$$\begin{aligned}
\int_{\mathbf{R}^N} f(x,u)w^+ dx \le \int_{\mathbf{R}^N} f(x,u)^+ w^+ dx &= \int_{\mathbf{R}^N} f(x,u^+)w^+ dx \\
&\le \int_{\mathbf{R}^N}\left(\xi|u^+| + K(\xi)|u^+|^{p-1}\right)w^+ dx \\
&\le \xi\|u^+\|_{L^2}\|w^+\|_{L^2} + K(\xi)\|u^+\|_{L^p}^{p-1}\|w^+\|_{L^p} \\
&\le \left(\frac{\delta}{2d}dist(u,P^-) + K_2 dist(u,P^-)^{p-1}\right)\|w^+\|,
\end{aligned}$$

with a constant $K_2 > 0$. So

$$dist\left(w, P^-\right)\left\|w^+\right\| \le \left(\frac{1}{2}dist\left(u, P^-\right) + \tilde{K}dist\left(u, P^-\right)^{p-1}\right)\left\|w^+\right\|,$$

with a constant $\tilde{K} > 0$. Thus

$$dist\left(w, P^-\right) \le \frac{1}{2}dist\left(u, P^-\right) + \tilde{K}dist\left(u, P^-\right)^{p-1}.$$

Hence, for $\varepsilon_0 > 0$ small enough

$$dist\left(-\frac{\mathbf{KG}(u)}{T(u)}, P^-\right) \le \frac{3}{4}dist\left(u, P^-\right)$$

for every $u \in P_\varepsilon^- \cap S_r$ with $0 < \varepsilon < \varepsilon_0$. In particular we have $A\left(\partial P_\varepsilon^-\right) \subset P_\varepsilon^-$. If moreover $u \in P_\varepsilon^-$ satisfies $A(u) = u$, then $u \in P^-$. If finally $u \neq 0$, we conclude $u(x) < 0$ for all x by the maximum principle [12]. Hence, every nontrivial solution $u \in P_\varepsilon^- \cap S_r$ of (1.1) is negative. Similarly, every nontrivial solution $u \in P_\varepsilon^+ \cap S_r$ of (1.1) is positive. □

In view of Proposition 4.2, the next proposition just follows from Liu and Sun [9]

Lemma 4.1 [9] Let E be a Hilbert space. Assume that $\Phi \in C^1(E, \mathbf{R})$, $\Phi'(u) = u - A(u)$ for $u \in E$, $D_1 \cap D_2 \neq \varnothing$, and $A(\partial D_i) \subset D_i \ (i = 1, 2)$. Then there is a p.g.v.f Q for Φ which enables D_1 and D_2 to be invariant sets of descending flow and $\partial D_i \subset C_H(D_i)(i = 1, 2)$.

Lemma 4.2 [9] Let E be a Hilbert space. Suppose $\Phi \in C^1(E, \mathbf{R})$ satisfies (*PS*) and $\Phi'(u)$ has the expression $\Phi'(u) = u - A(u)$ for $u \in E$. Assume that D_1 and D_2 are open convex subset of E with the properties that $D_1 \cap D_2 \neq \varnothing$ and $A(\partial D_i) \subset D_i \ (i = 1, 2)$. If there exists a path $h : [0,1] \to E$ such that

$$h(0) \in D_1 \setminus D_2, h(1) \in D_2 \setminus D_1,$$

and

$$\inf_{u \in D_1 \cap D_2} \Phi(u) > \inf_{t \in [0,1]} \Phi(h(t)).$$

Then Φ has at least four critical points, one in $D_1 \cap D_2$, one in $D_1 \setminus \overline{D_2}$, one in $D_2 \setminus \overline{D_1}$, and one in $E \setminus \left(\overline{D_1} \cup \overline{D_2}\right)$.

Note that I' is a p.g.v.f of I such that P_ε^+ and P_ε^- are invariant for the associated descending flow. Moreover

$$\partial P_\varepsilon^\pm \subset \mathscr{S}\left(P_\varepsilon^\pm\right) \ \text{ for } 0 < \varepsilon \le \varepsilon_0. \tag{4.9}$$

holds following from Proposition 4.2 and Lemma 4.1.

Proposition 4.3 If $0 < \varepsilon \le \varepsilon_0$, then $\overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \subset \mathscr{S}_0$. In particular there holds

$$J(u) > -\frac{1}{2}\|u\|^2 \ \text{ for } u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \setminus \{0\}. \tag{4.10}$$

Proof. First, from (A_2) and (A_3) we have

$$\int_{\mathbf{R}^N} F(x,u)\mathrm{d}x \le \frac{1}{\eta}\int_{\mathbf{R}^N} f(x,u)u\mathrm{d}x \le \frac{C}{\eta}\left(\|u\|_{L^2}^2 + \|u\|_{L^p}^p\right).$$

Since by (3.3) we infer that

$$\|u\|_{L^s}^2 = \|u^+ + u^-\|_{L^s}^2 = \|u^+\|_{L^s}^2 + \|u^-\|_{L^s}^2$$
$$\le C_s^2 dist\left(u, P^-\right)^2 + C_s^2 dist\left(u, P^+\right)^2 \le 2\varepsilon^2 C_s^2.$$

Hence, for $u \in P_\varepsilon^+ \cap P_\varepsilon^-$

$$J(u) = \frac{1}{2}\|u\|^2 - \int_{\mathbf{R}^N} F(x,u)\,\mathrm{d}x - \frac{1}{2}\|u\|^2 \ge \frac{1}{2}\|u\|^2 - \frac{C}{\eta}\left(\|u\|_{L^2}^2 + \|u\|_{L^p}^p\right) - \frac{1}{2}\|u\|^2$$
$$\ge \frac{1}{2}\|u\|^2 - \frac{C}{\eta}\left(2\varepsilon^2 C_s^2 + \left(2\varepsilon^2 C_s^2\right)^{\frac{p}{2}}\right) - \frac{1}{2}\|u\|^2 \ge -\frac{1}{2}\|u\|^2.$$

Next we recall that $P_\varepsilon^+ \cap P_\varepsilon^-$ contains no critical points of I by Proposition 4.2. Thus by (4.9), (4.10) and the invariance of $P_\varepsilon^+ \cap P_\varepsilon^-$ we find that

$$\sigma(t,u) \to 0 \text{ as } t \to \omega_+(u)$$

for every $u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-}$ as claimed. □

As a consequence of the preceding discussion, the existence of three solutions with one changing sign follows from Lemma 4.2.

Proof of Theorem 1.1 Choose $v \in S_r$, and set

$$C := \left\{tv^+ + sv^- : t \ge 0, s \ge 0\right\}.$$

It follow from Proposition 1, there exists $R > 0$ such that

$$J(u) < -\frac{1}{2}\|u\|^2 \text{ for } u \in C \setminus B_R(0).$$

By the choose of *r*, we have $r > R$, now we define the path

$$h:[0,1] \to S_r,\ h(t) = t\frac{r}{\|v^+\|}v^+ + (1-t)\frac{r}{\|v^-\|}v^-,$$

then

$$h(0) = \frac{r}{\|v^-\|}v^- \in C \cap P_\varepsilon^- \cap S_r,\quad h(1) = \frac{r}{\|v^+\|}v^+ \in C \cap P_\varepsilon^+ \cap S_r,$$

hence

$$\sup_{t\in[0,1]} I(h(t)) = \sup_{t\in[0,1]} J(h(t)) < -\frac{r^2}{2}.$$

Applying Proposition 4.3, we obtain

$$J(u) > -\frac{1}{2}\|u\|^2 \text{ for } u \in \overline{P_\varepsilon^+} \cap \overline{P_\varepsilon^-} \setminus \{0\}.$$

Thus

$$\inf_{u\in \overline{P_\varepsilon^+ \cap P_\varepsilon^-}} I(u) > -\frac{r^2}{2}.$$

So we have

$$\inf_{u\in \overline{P_\varepsilon^+ \cap P_\varepsilon^-}} I(u) > \sup_{t\in[0,1]} I(h(t)).$$

Since I satisfies $(PS)^-$, $I'(u) = u - A(u)$ for $u \in S_r$, P_ε^+ and P_ε^- are open convex subsets of H, $P_\varepsilon^+ \cap P_\varepsilon^- \cap S_r \neq \varnothing$. And by Proposition 4.2, $A\left(\overline{P_\varepsilon^\pm} \cap S_r\right) \subset \overline{P_\varepsilon^\pm} \cap S_r$. It follows from Lemma 4.2 that I has a critical points $u_+ \in \left(P_\varepsilon^+ \setminus \overline{P_\varepsilon^-}\right) \cap S_r$, $u_- \in \left(P_\varepsilon^- \setminus \overline{P_\varepsilon^+}\right) \cap S_r$, $\overline{u} \in S_r \setminus \left(\overline{P_\varepsilon^+} \cup \overline{P_\varepsilon^-}\right)$, where u_+ is a positive solution of (1.1), u_- is a negative solution of (1.1) and $\overline{u}$ is a sign changing solution of (1.1). □

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Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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