Scientific Research Publishing

Applied Mathematics, 2022, 13, 602-611
https://www.scirp.org/journal/am
ISSN Online: 2152-7393
ISSN Print: 2152-7385

Goal Achieving Probabilities of Mean-Variance Strategies in a Market with Regime-Switching Volatility

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How to cite this paper: Ferland, R. and Watier, F. (2022) Goal Achieving Probabilities of Mean-Variance Strategies in a Market with Regime-Switching Volatility. *Applied Mathematics*, **13**, 602-611.
https://doi.org/10.4236/am.2022.137038

Received: May 31, 2022
Accepted: July 16, 2022
Published: July 19, 2022

Abstract

In this paper, we establish properties for the switch-when-safe mean-variance strategies in the context of a Black-Scholes market model with stochastic volatility processes driven by a continuous-time Markov chain with a finite number of states. More precisely, expressions for the goal-achieving probabilities of the terminal wealth are obtained and numerical comparisons of lower bounds for these probabilities are shown for various market parameters. We conclude with asymptotic results when the Markovian changes in the volatility parameters appear with either higher or lower frequencies.

Keywords

First Passage Time Probabilities, Mean-Variance Strategy, Regime-Switching Model

1. Introduction

In the financial world, an investor is routinely subjected to finding strategies that offer higher returns with reduced risks. In his seminal paper [1], Nobel prize laureate Markowitz introduced the myopic (single period) mean-variance portfolio management problem where one calibrates the amount of wealth invested in risky assets (stocks) and a riskless asset (bond) in such a way that it minimizes the variance of a terminal wealth while targeting an average end return. Since then, scores of innovative research problems arose related to his original static model as well as dynamic extensions in both discrete and continuous time, as seen for example in the following recent papers: [2] [3] [4].

It's worth noting that since the unconstrained mean-variance approach is solely based on averaged return, then an investor might experience undesired marked scenarios such as returns below a safe investment in a bank account with guaranteed interest rate or even worst events such as bankruptcy. In an effort to reduce the probability of encountering these undesired scenarios while still aiming for the target wealth at the end of the investment horizon, Zhou and Li [5] devised a hybrid strategy that we will call here the switch-when-safe strategy. More precisely, in a continuous-time setting under a Black-Scholes market model with deterministic parameters, the investor follows the optimal unconstrained mean-variance strategy up to the first (random) moment, if it occurs, where he could reinvest all of his cumulative wealth in a riskless asset so that it would generate the desired wealth at the end of the investment horizon. In their paper, they discovered the following astonishing properties:

- The goal-achieving probability depends on neither the initial wealth nor the desired terminal wealth;
- The goal-achieving probability has an explicit expression in terms of market parameters and time horizon;
- The goal-achieving probability has a universal lower bound of 0.80, which depend on neither the market parameters nor the time horizon.

Still, in the context of deterministic market parameters in a Black Scholes model with stock prices driven by Brownian motions, these same properties were also uncovered when one considers cone-constrained mean-variance strategies such as no short-selling strategies [6] [7]. In this paper, we wish to explore if these properties carry on to more general market models for example by considering a Black-Scholes model with added randomness, more precisely, while maintaining deterministic interest for the riskless asset and deterministic drift parameters for the risky asset, we will allow the volatility parameter of the risky asset to change, depending on the state of a continuous-time Markov chain, independent of the stock prices driven by Brownian motions.

2. Market Model and Regime-Switching Mean-Variance Strategy

The market model is composed of a riskless asset and m risky assets with a volatility matrix $\sigma_{\alpha(t)}$ depending of an independent Markov chain α. The price $S_0(t)$ of the riskless asset a time t follow the dynamics given by the ODE:

$$\mathrm{d}S_0(t) = r(t)S_0(t)\mathrm{d}t$$

while the price of the risky assets follow the dynamics given by the SDEs:

$$\mathrm{d}S_i(t) = S_i(t)\left[\mu_i(t)\mathrm{d}t + \sum_{j=1}^{m}\sigma_{ij,\alpha(t)}\mathrm{d}W_j(t)\right], \quad i = 1,\cdots,m$$

where W_j are independent standard Brownian motions and $\{\alpha(t): t \geq 0\}$ is a continuous-time Markov chain with a finite set of states $\{1,\cdots,S\}$.

Let $W(t) = [W_j(t)]_{m\times 1}$, $\sigma_{\alpha(t)} = [\sigma_{ij,\alpha(t)}]_{m\times m}$, $B(t) = [\mu_i(t) - r(t)]_{1\times m}$ and $\pi(t) = [\pi_i(t)]_{m\times 1}$ be the investor portfolio: $\pi_i(t)$ is the amount invested in the i^{th} stock at time t. Then the self-financing wealth process X of the investor is driven by the SDE

$$\mathrm{d}X(t) = \left[r(t)X(t) + B(t)\pi(t)\right]\mathrm{d}t + \pi(t)\sigma_{\alpha(t)}(t)\mathrm{d}W(t), \quad X(0) = x_0.$$

A *mean-variance strategy* $\pi_{MV}(t)$ is one that minimizes the variance of the terminal wealth $Var(X(T))$ under the constraint that the expected terminal wealth satisfies $E(X(T)) = z$ where $z > x_0 \mathrm{e}^{\int_0^T r(s)\mathrm{d}s}$.

Zhou and Yin [8] showed that, for a regime-switching volatility model, this optimal strategy is given by

$$\pi_{MV}(t) = -\left[\sigma_{\alpha(t)}(t)\sigma_{\alpha(t)}(t)\right]^{-1} B(t)\left[X(t) + \lambda \mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}\right]$$

where the Lagrange multiplier λ is given by

$$\lambda = \frac{z - x_0 P(0,\alpha(0))\mathrm{e}^{-\int_0^T r(s)\mathrm{d}s}}{P(0,\alpha(0))\mathrm{e}^{-2\int_0^T r(s)\mathrm{d}s} - 1}$$

and $P(t,k)$ is the solution to the following ODE system

$$\frac{\partial P(t,k)}{\partial t} = \left[\theta_k(t)\theta_k(t) - 2r(t)\right]P(t,k) - \sum_{\ell=1}^{S} q_{k\ell} P(t,\ell)$$

$$P(T,k) = 1$$

where $\theta_k(t) = B(t)\sigma_k^{-1}$ and $Q = [q_{k\ell}]_{S\times S}$ is the infinitesimal generator of Markov chain $\{\alpha(t): t \geq 0\}$.

Consequently, following Itô's formula, the wealth process $X_{MV}(t)$ of the mean-variance startegy can be expressed as

$$X_{MV}(t) = \left(x_0 \mathrm{e}^{\int_0^T r(s)\mathrm{d}s} + \lambda\right)Z(t) - \lambda \mathrm{e}^{-\int_t^T r(s)\mathrm{d}s}$$

where

$$Z(t) = \exp\left\{-\frac{3}{2}\int_0^t \left\|\theta_{\alpha(s)}(s)\right\|^2 \mathrm{d}s - \int_0^t \theta_{\alpha(s)}(s)\mathrm{d}W(s)\right\}.$$

This form is well-suited to the computations in the next section.

3. Switch-When-Safe Mean-Variance Strategy and Goal Achieving Probabilities

Consider the following stopping time:

$$\tau_z = \inf\left\{0 \leq t \leq T : X_{MV}(t)\mathrm{e}^{\int_t^T r(s)\mathrm{d}s} = z\right\}.$$

This time, if it exists, is the first moment at which the wealth is such that, invested in the riskless asset, it would have a final value equal to the targeted expected terminal wealth of the mean-variance strategy.

The *switch-when-safe mean-variance strategy* of [5] is defined as

$$\pi_{SWS}(t) = \begin{cases} \pi_{MV}(t) & \text{if } t \leq \tau_z \wedge T, \\ 0 & \text{otherwise.} \end{cases}$$

Now observe that

$$X_{MV}(t) - z\mathrm{e}^{-\int_t^T r(s)\mathrm{d}s} = \frac{x_0\mathrm{e}^{\int_0^T r(s)\mathrm{d}s} - z}{1 - P(0,\alpha(0))\mathrm{e}^{-2\int_0^T r(s)\mathrm{d}s}}\left[Z(t) - P(0,\alpha(0))\mathrm{e}^{-2\int_0^T r(s)\mathrm{d}s}\right].$$

Since $x_0\mathrm{e}^{\int_0^T r(s)\mathrm{d}s} - z < 0$, it follows that the equality $X_{MV}(t)\mathrm{e}^{\int_t^T r(s)\mathrm{d}s} = z$ is verified if and only if

$$\frac{3}{2}\int_0^t \left\|\theta_{\alpha(s)}(s)\right\|^2 \mathrm{d}s + \int_0^t \theta_{\alpha(s)}(s)\mathrm{d}W(s) = 2\int_0^T r(s)\mathrm{d}s - \ln P(0,\alpha(0)).$$

From this condition, we see that, as it is the case in [5], the goal-achieving probability of the switch-when-safe mean-variance strategy for a regime-switching volatility model does not depend on either the initial wealth or the desired terminal wealth.

Now let us find an expression for the goal-achieving probabilities in the case of a model with one risky asset that is W is reduced to a one-dimensional brownian motion, let

$$Y(t) = \int_0^t \frac{3}{2}\theta^2_{\alpha(s)}(s)\mathrm{d}s + \int_0^t \theta_{\alpha(s)}(s)\mathrm{d}W(s)$$

First, according to Buffington and Elliott [9], the characteristic function of the diffusion process Y is given by

$$\phi_t(u) = \left\langle \exp\left(Qt + \begin{bmatrix} \left(\frac{3}{2}iu - \frac{1}{2}u^2\right)\theta_1^2 & 0 & \cdots \\ 0 & \cdots & 0 \\ \cdots & 0 & \left(\frac{3}{2}iu - \frac{1}{2}u^2\right)\theta_S^2 \end{bmatrix} t\right)\pi(0), 1_S \right\rangle$$

where 1_S is a S-dimensional vector of ones.

Let $a = 2\int_0^T r(s)\mathrm{d}s - \ln P(0,1)$ represent the barrier, if T_a is the stopping time defined by

$$T_a = \inf\{0 \le t \le T : Y(t) = a\}$$

then

$$\Pr(\tau_z \le T) = \Pr(T_a \le T).$$

By introducing the Wiener-Hopf factorization of the process $(Y(t),\alpha(t))$ that is to say, the couple (Q_+, Q_-) which solves for every $u > 0$

$$\Xi(-Q_+) = \Xi(Q_-) = 0$$

where

$$\Xi(P) = \frac{1}{2}\Sigma^2 P^2 + VP + Q - uI_S$$

with $\Sigma = diag(\theta_1,\cdots,\theta_S)$, $V = diag\left(\frac{3}{2}\theta_1^2,\cdots,\frac{3}{2}\theta_S^2\right)$, Q the infinitesimal generator and I_S the $S \times S$ identity matrix, then, following Jiang and Pistorius [10], the associated Laplace transform Ψ_a of the random variable T_a is given by

$$\Psi_a = \pi(0)\exp(aQ_+)1_S$$

therefore, through Laplace transform inversion, we deduce

$$\Pr(T_a \le T) = \frac{\Psi_a(0)}{2} - \frac{1}{\pi}\mathrm{Re}\left(\int_0^\infty \mathrm{e}^{-iu}\frac{\Psi_a(-iu)}{iu}\mathrm{d}u\right).$$

Moreover, since the ratio $\frac{\frac{3}{2}\theta_i^2}{(\theta_i)^2} = \frac{3}{2}$ is constant for $i = 1,\cdots,S$ then according to Hieber [11] the last expression is reduced to

$$\Pr(T_a \le T) = \frac{1+\exp(3a)}{2} - \frac{1}{\pi}\mathrm{Re}\left(\int_0^\infty \frac{\exp(3a+iua)-\exp(-iua)}{iu}\phi_T(u)\mathrm{d}u\right).$$

Both expressions can easily be evaluated numerically. However, it is worth mentioning that, even if one could find explicit forms for the exponential matrices (which is the case for $S = 2$ for example) appearing in these expressions, searching for possible closed-formed formulas for the integrals involved could prove to be quite challenging.

One notable exception is the trivial case where all possible values of the volatility matrix are reduced to a single constant matrix. Then we have $\theta_{\alpha(t)} \equiv \theta$ that is $Y(t)$ revert to a standard Brownian motion with drift and according to [5]:

$$\Pr(T_a \le T) = \Phi\left(\frac{1}{2}\|\theta\|\sqrt{T}\right) + \mathrm{e}^{3\|\theta\|^2 T}\Phi\left(-\frac{5}{2}\|\theta\|\sqrt{T}\right).$$

Figure 1 shows the probabilities in this case as a function of $x = \|\theta\|\sqrt{T}$.

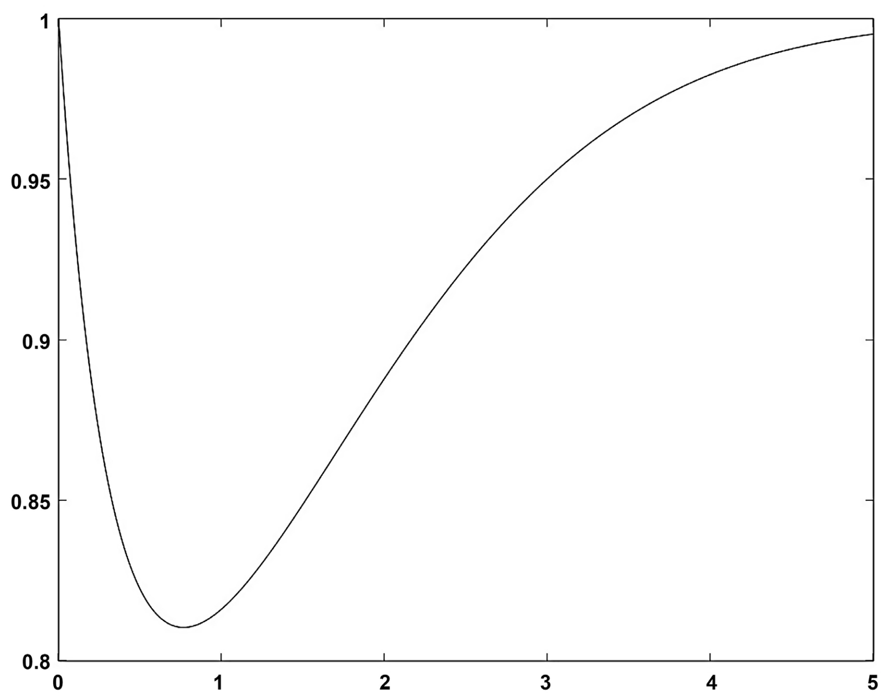

Figure 1. Goal-achieving probabilities as a function of $x = \|\theta\|\sqrt{T}$.

For the numerical study of lower bound probabilities, we will suppose hereon that we have one risky asset, the parameters r and μ are constant, and the volatility parameter σ follows a 2-state continuous-time Markov chain with an infinitesimal generator Q taking the form

$$Q = \begin{bmatrix} q_{11} & -q_{11} \\ -q_{22} & q_{22} \end{bmatrix}$$

where $q_{11}, q_{22} < 0$. In this case, the constant interest r allows us to have the explicit solution to the ODE system

$$\begin{bmatrix} P(t,1) \\ P(t,2) \end{bmatrix} = \exp\left(-(T-t)(M-Q)\right)\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

with

$$M = \begin{bmatrix} \theta_1^2 - 2r & 0 \\ 0 & \theta_2^2 - 2r \end{bmatrix}.$$

Furthermore, since α is 2-state Markov chain, the matrix Q_+ can be written explicitly [11] as

$$Q_+ = \begin{bmatrix} \dfrac{-\beta_{3,u}\beta_{4,u} + 2(q_{11} - u)/\theta_1^2}{\beta_{3,u} + \beta_{4,u} + 3} & \dfrac{-2q_{11}/\theta_1^2}{\beta_{3,u} + \beta_{4,u} + 3} \\ \dfrac{-2q_{22}/\theta_2^2}{\beta_{3,u} + \beta_{4,u} + 3} & \dfrac{-\beta_{3,u}\beta_{4,u} + 2(q_{22} - u)/\theta_2^2}{\beta_{3,u} + \beta_{4,u} + 3} \end{bmatrix}$$

where $\beta_{3,u} < \beta_{4,u}$ are the real positive roots of the quartic equation

$$\left(\frac{1}{2}\theta_1^2\beta^2 + \frac{3}{2}\theta_1^2\beta + q_{11} - u\right)\left(\frac{1}{2}\theta_2^2\beta^2 + \frac{3}{2}\theta_2^2\beta + q_{22} - u\right) - q_{11}q_{22} = 0.$$

Following the Cayley-Hamilton theorem we then have

$$\exp(aQ_+) = \left(\frac{\beta_{3,u}\mathrm{e}^{-a\beta_{4,u}} - \beta_{4,u}\mathrm{e}^{-a\beta_{3,u}}}{\beta_{3,u} - \beta_{4,u}}\right)I_2 + \left(\frac{\mathrm{e}^{-a\beta_{3,u}} - \mathrm{e}^{-a\beta_{4,u}}}{\beta_{3,u} - \beta_{4,u}}\right)Q_+$$

which leads us to an explicit expression for the Laplace transform Ψ_a. We will use it in our numerical computation of the goal-achieving probabilities $P(T_a \le T)$.

As an example, consider a market model with a single asset and a two-state volatility:

$$\mu = 0.10, \quad r = 0.01, \quad Q = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

We can compute the goal-achieving probabilities $P(T_a \le T)$ like in **Figure 1** and find a lower bound for them as a function of the different values of the stock's regime-switching volatility. **Table 1** gives the lower bound probabilities, assuming the initial regime-switching state is $\alpha(0) = 1$.

Clearly one observes that, in presence of a true regime-switching volatility model ($\sigma_1 \ne \sigma_2$), the lower bound probabilities cross the threshold of its deterministic model counterpart. Moreover, as σ_1 takes on larger values while σ_2 takes on lower values the lower bound probabilites gets fairly small, for example

Table 1. Lower bounds of goal-achieving probabilities.

σ_1/σ_2	0.05	0.10	0.15	0.20	0.25
0.05	0.810	0.795	0.783	0.778	0.776
0.10	0.771	0.810	0.800	0.788	0.777
0.15	0.706	0.797	0.810	0.806	0.800
0.20	0.645	0.776	0.806	0.810	0.808
0.25	0.594	0.753	0.798	0.808	0.810

if we take $\sigma_1 = 0.50$ and $\sigma_1 = 0.01$ the lower bound probability decreases to a mere 0.166.

4. Limit Cases of Goal Achieving Probabilities

Assume now that Q depends on a parameter k:

$$Q(k) = k\begin{bmatrix} q_{11} & -q_{11} \\ -q_{22} & q_{22} \end{bmatrix}$$

with $k > 0$. We will study the first passage time probabilities when either $k \uparrow \infty$ or $k \downarrow 0$, which corresponds respectively to the case where the regime-switching jumps appear with high frequency or are scarce.

- $k \uparrow \infty$ (average time to next jump tends to zero)

$$\lim_{k\to\infty} \exp\left(-T\left(M-Q\right)\right) = \begin{bmatrix} \dfrac{q_{22}}{q_{11}+q_{22}} & \dfrac{q_{11}}{q_{11}+q_{22}} \\ \dfrac{q_{22}}{q_{11}+q_{22}} & \dfrac{q_{11}}{q_{11}+q_{22}} \end{bmatrix} \mathrm{e}^{-\left(\frac{q_{22}}{q_{11}+q_{22}}M_{11}+\frac{q_{11}}{q_{11}+q_{22}}M_{22}\right)T}$$

and therefore

$$\begin{bmatrix} P(0,1) \\ P(0,2) \end{bmatrix} \to \begin{bmatrix} \mathrm{e}^{-\left(\frac{q_{22}}{q_{11}+q_{22}}M_{11}+\frac{q_{11}}{q_{11}+q_{22}}M_{22}\right)T} \\ \mathrm{e}^{-\left(\frac{q_{22}}{q_{11}+q_{22}}M_{11}+\frac{q_{11}}{q_{11}+q_{22}}M_{22}\right)T} \end{bmatrix}.$$

The barrier $a \to 2rT - \ln P\left(0,\alpha(0)\right) = \theta_\infty^{(2)}T$ where

$$\theta_\infty^{(2)} = \frac{q_{22}}{q_{11}+q_{22}}\theta_1^2 + \frac{q_{11}}{q_{11}+q_{22}}\theta_2^2 .$$

We can also show after tedious calculations that

$$\lim_{k\to\infty} \Psi_a = \mathrm{e}^{\frac{\lambda}{\nu}\left(1-\sqrt{1+\frac{2\nu^2}{\lambda}u}\right)}$$

where $\nu = \dfrac{2}{3}T$ and $\lambda = \theta_\infty^{(2)}T^2$.

This expression corresponds to the Laplace transform of the inverse Gaussian (or Wald) density with mean ν and shape parameter λ, therefore

$$\Pr\left(T_a \le T\right) \to \Phi\left(\sqrt{\frac{\lambda}{T}}\left(\frac{T}{\nu}-1\right)\right) + \mathrm{e}^{\frac{2\lambda}{\nu}}\Phi\left(-\sqrt{\frac{\lambda}{T}}\left(\frac{T}{\nu}+1\right)\right)$$

$$= \Phi\left(\frac{1}{2}\sqrt{\theta_\infty^{(2)}T}\right) + e^{3\theta_\infty^{(2)}T}\Phi\left(-\frac{5}{2}\sqrt{\theta_\infty^{(2)}T}\right)$$

For the single asset model of the previous section, with regime-switching volatilities $\sigma_1 = 0.10$ and $\sigma_2 = 0.20$, **Figure 2** below shows the goal-achieving probabilites for increasing values of *k*.

- $k \downarrow 0$ (average time to next jump tends to infinity)

$$\lim_{k\to 0^+} \exp(-T(M-Q)) = \exp(-TM)$$

and therefore

$$\begin{bmatrix} P(0,1) \\ P(0,2) \end{bmatrix} \to \begin{bmatrix} e^{-M_{11}T} \\ e^{-M_{22}T} \end{bmatrix}.$$

The barrier $a \to 2rT - \ln P(0,\alpha(0)) = \theta_{\alpha(0)}T$.

In this case, we can obtain the limit of the passage-time probability in a straightforward manner. Since the average time to the next jumps tends towards infinity, the Markov chain $\alpha(t)$ will have a tendency to stay at its intital state $\alpha(0)$, thus

$$Y(t) \to \frac{3}{2}\theta_{\alpha(0)}^2 t + \theta_{\alpha(0)}W(t)$$

and therefore

$$\Pr(T_a \le T) \to \Phi\left(\frac{1}{2}\theta_{\alpha(0)}\sqrt{T}\right) + e^{3\theta_{\alpha(0)}^2 T}\Phi\left(-\frac{5}{2}\theta_{\alpha(0)}\sqrt{T}\right).$$

For the same example as above, **Figure 3** illustrates this result for decreasing values of *k*.

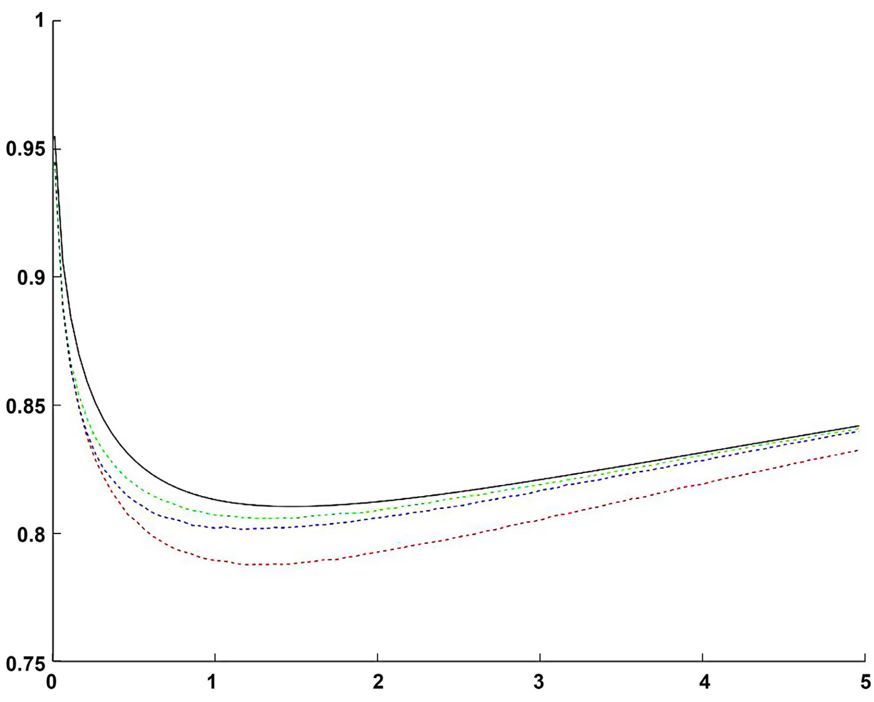

Figure 2. Goal-achieving probabilities for increasing *k*.

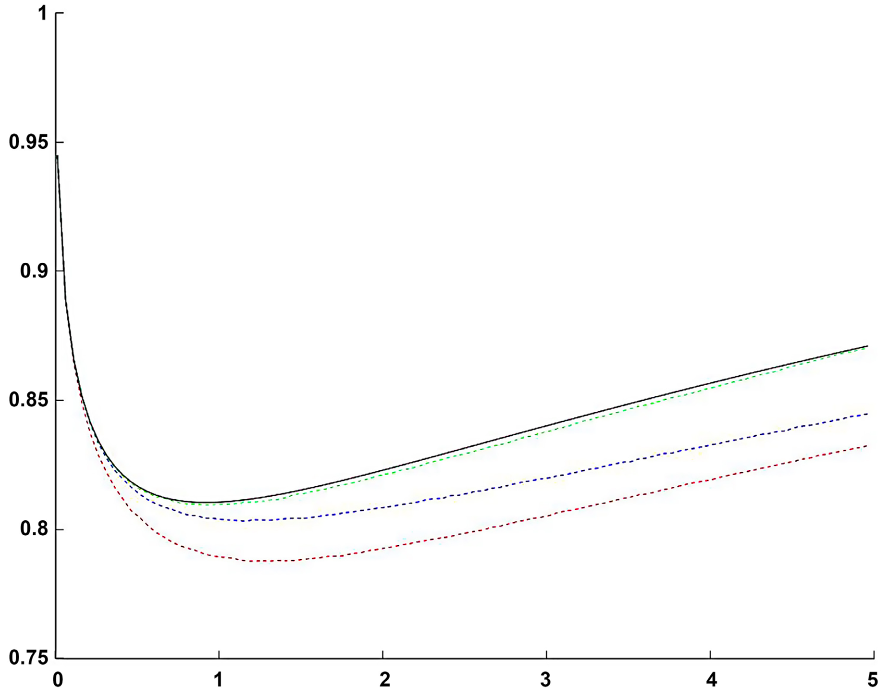

Figure 3. Goal-achieving probabilities for decreasing *k*.

5. Conclusion

In the context of a Black-Scholes market model with stochastic volatility processes driven by a continuous-time Markov chain with finite states, we obtained tractable expressions for the goal-achieving probabilities of switch-when-safe strategies as first introduced by Zhou and Li [5]. We observed that the goal-achieving probabilities are independent of the value of the initial wealth and targeted terminal mean wealth, a property shared with the standard Black-Scholes market counterpart. Unfortunately, it appears that a universal lower bound for these probabilities does not exist for the set of all possible market parameters and infinitesimal generators of the Markovian process as illustrated by our numerical studies. Finally, when the Markovian regime is allowed to either attain higher or lower frequencies than the first-passage time probabilities expressions converge to closed-form formulas.

Acknowledgements

The second author sincerely thanks the Natural Sciences and Engineering Research Council of Canada (NSERC) for its support.

Conflicts of Interest

The authors declare no conflicts of interest regarding the publication of this paper.

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